

# VERMA-TYPE MODULES FOR QUANTUM AFFINE LIE ALGEBRAS

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**ABSTRACT.** Let  $\mathfrak{g}$  be an untwisted affine Kac-Moody algebra and  $M_J(\lambda)$  a Verma-type module for  $\mathfrak{g}$  with  $J$ -highest weight  $\lambda \in P$ . We construct quantum Verma-type modules  $M_J^q(\lambda)$  over the quantum group  $U_q(\mathfrak{g})$ , investigate their properties and show that  $M_J^q(\lambda)$  is a true quantum deformation of  $M_J(\lambda)$  in the sense that the weight structure is preserved under the deformation. We also analyze the submodule structure of quantum Verma-type modules.

## Introduction.

The representation theory of Kac-Moody algebras is much richer than that of finite-dimensional simple algebras. In particular, Kac-Moody algebras have modules containing both finite and infinite-dimensional weight spaces, something that cannot happen in the finite-dimensional setting [Le]. These representations of Kac-Moody algebras arise from taking non-standard partitions of the root system, partitions which are not Weyl-equivalent to the standard partition into positive and negative roots. For affine algebras, there is always a finite number of Weyl-equivalency classes of these nonstandard partitions. Corresponding to each partition is a Borel subalgebra and one can form representations induced from one-dimensional modules for these nonstandard Borel subalgebras. These induced modules are called modules of Verma-type, in analogy with standard Verma modules induced from a standard Borel subalgebra.

Verma-type modules were first studied and classified by Jakobsen and Kac [JK1, JK2], and by Futorny [Fu1, Fu2]. Further work elucidating their structure, including the construction of the appropriate categorical setting, determination of irreducibility criteria, submodule structure, BGG duality and BGG resolutions can be found in [Co1, Co2, CFM, Fu4, Fu5, and references therein].

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The theory of Verma-type modules is best developed in the case where the central element of  $\mathfrak{g}$  acts with non-zero charge. In this case we say the level is non-zero. Verma-type modules of zero level are still not completely classified. There are many technical difficulties, for example, a Verma-type module of level zero may have subquotients that are not quotients of Verma-type modules [Fu3, Proposition 5(ii)]. As the affine theory is not complete, we restrict our attention on structural questions to representations of non-zero level, except in the case of imaginary Verma modules, where the level zero theory is more complete. However, the techniques we use can be generalized (with more technicalities) to the case of level zero where  $\lambda$  is in “general position” (see [Fu4] for details of the affine theory).

Since their introduction by Drinfeld [Dr] and Jimbo [Ji] in 1985, there has been a tremendous interest in studying quantized enveloping algebras for Kac-Moody algebras and their representations. Quantum groups have turned out to be extremely important objects with rich and diverse connections to an ever-increasing number of areas of mathematics and physics. A vigorous body of research is developing on determining their structure, their representations and their applications.

In many cases, the representation theory of quantum groups parallels that of the associated underlying classical algebras, although often with some subtle differences. The closest parallel comes when the classical and quantum representations have the same weight structure. In this paper, we construct quantum Verma-type modules over the quantum group  $U_q(\mathfrak{g})$  associated to an untwisted affine Kac-Moody algebra  $\mathfrak{g}$ , prove they have the same weight structure as the corresponding Verma-type modules, and investigate some of their properties.

One of the problems to be faced when studying nonstandard representations of a Kac-Moody algebra  $\mathfrak{g}$ , is that the absence of a general PBW theorem for quantum groups means that we cannot lift the triangular decomposition of  $\mathfrak{g}$  up to a triangular decomposition of  $U_q(\mathfrak{g})$ . In [CFKM], the authors studied the simplest non-standard case, that of Verma-type modules for the algebra  $U_q(A_1^{(1)})$ . In the case of  $U_q(A_1^{(1)})$ , there is (up to Weyl-equivalence) only one non-standard partition and the associated representations are called imaginary Verma modules. In [CFKM], the authors had to construct an appropriate PBW basis to deal with that particular case (see their Proposition 2.2). The techniques used there do not easily generalize to the case of all affine algebras.

In this paper we rely heavily on the work by Beck [Be1, Be2] and Beck and Kac [BK] on PBW bases for quantum groups of affine algebras, and we exploit a particularly convenient description of the nonstandard partition of the root system used to construct the quantum Verma-type modules. This approach leads to one of our key ideas, that it is possible to determine a basis for the quantum Verma-type modules in a unified manner without giving a PBW basis for the algebra.

Having constructed the quantum representations, we check that they are true quantum deformations of the equivalent classical modules. That is, the quantum and classical modules have the same weight structure. To do this, we follow the  $\mathbb{A}$ -form technique originally introduced by Lusztig [Lu], and subsequently refined and developed by Kang and co-authors [Ka, CFKM, BKMe]. For an overview of this procedure and a summary of known quantum deformation results, see [M]. Our main result of this section, generalizing that of [CFKM], is that any quantum Verma-type module with integral  $J$ -highest weight  $\lambda$  is a quantum deformation of the equivalent Verma-type module over  $U(\mathfrak{g})$  for  $\mathfrak{g}$  an untwisted affine Kac-Moody

algebra.

Using the quantum deformation theorem, we study some of the structural properties of quantum Verma-type modules. In particular, we prove a general irreducibility criterion for quantum Verma-type modules and probe the structure of quantum imaginary Verma modules at level zero, when they are reducible. The results obtained are similar to those given for non-quantum imaginary Verma modules in [Fu3], showing that these quantum modules are closely related to their classical cousins. Some of the results on quantum imaginary Verma modules obtained in this paper were announced in [FGM].

The structure of the paper is as follows. First, we recall background information and establish notation in Section 1. In Section 2 we review the construction of Verma-type modules for affine algebras. In Section 3 we construct quantum Verma-type modules and provide them with a basis. Section 4 constructs  $\mathbb{A}$ -forms, and Section 5 gives the classical limits and quantum deformation theorem. Section 6 discusses the structural results for Verma-type modules in general and Section 7 considers quantum imaginary Verma modules at level zero. For additional basic background material and notation on Kac-Moody algebras, see the book by Kac [K]; for background information on quantum groups, see the excellent texts by Chari and Pressley [CP] and Jantzen [Ja].

## 1. Preliminaries.

**1.1.** Let  $N$  be a positive integer. Fix index sets  $\dot{I} = \{1, \dots, N\}$  and  $I = \{0, \dots, N\}$ . Let  $\dot{\mathfrak{g}}$  be a finite-dimensional simple complex Lie algebra with Cartan subalgebra  $\dot{\mathfrak{h}}$ , root system  $\dot{\Delta} \subset \dot{\mathfrak{h}}^*$ , and set of simple roots  $\dot{\Pi} = \{\alpha_1, \dots, \alpha_N\}$ . Denote by  $\dot{\Delta}_+$  and  $\dot{\Delta}_-$ , the positive and negative roots of  $\dot{\mathfrak{g}}$ . Let  $\dot{Q} = \oplus_{i=1}^N \mathbb{Z}\alpha_i$  be the root lattice of  $\dot{\mathfrak{g}}$ , and let  $\dot{A} = (a_{ij})_{1 \leq i, j \leq N}$  be the Cartan matrix for  $\dot{\mathfrak{g}}$ . Define a basis  $h_1, \dots, h_N$  of  $\dot{\mathfrak{h}}$  by  $\alpha_i(h_j) = a_{ij}$ . Let  $\dot{P} = \{\lambda \in \dot{\mathfrak{h}}^* \mid \lambda(h_i) \in \mathbb{Z}, i = 1, \dots, N\}$  be the weight lattice of  $\dot{\mathfrak{g}}$ . Let  $(\cdot | \cdot)$  denote both the symmetric invariant bilinear form on  $\dot{\mathfrak{g}}$  and the induced form on  $\dot{\mathfrak{g}}^*$ , normalized so that  $(\alpha | \alpha) = 2$  for any short root  $\alpha$ . For  $i = 1, \dots, N$ , let  $d_i = (\alpha_i | \alpha_i)/2$ . Then each  $d_i$  is a positive integer, the  $d_i$  are relatively prime, and the diagonal matrix  $\dot{D} = \text{diag}(d_1, \dots, d_N)$  is such that  $\dot{D}\dot{A}$  is symmetric.

**1.2.** Let  $\mathfrak{g}$  denote the untwisted affine Kac-Moody algebra associated to  $\dot{\mathfrak{g}}$ . Then  $\mathfrak{g}$  has the loop space realization

$$\mathfrak{g} = \dot{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where  $c$  is central in  $\mathfrak{g}$ ;  $d$  is the degree derivation, so that  $[d, x \otimes t^n] = nx \otimes t^n$  for any  $x \in \dot{\mathfrak{g}}$  and  $n \in \mathbb{Z}$ , and  $[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + \delta_{n+m,0}n(x|y)c$  for all  $x, y \in \dot{\mathfrak{g}}$ ,  $n, m \in \mathbb{Z}$ . We set  $\mathfrak{h} = \dot{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ .

The algebra  $\mathfrak{g}$  has a Cartan matrix  $A = (a_{ij})_{0 \leq i, j \leq N}$  which is an extension of  $\dot{A}$ . There exists an integer  $d_0$  and a diagonal matrix  $D = \text{diag}(d_0, \dots, d_N)$  such that  $DA$  is symmetric. An alternative Chevalley-Serre presentation of  $\mathfrak{g}$  is given by defining it as the Lie algebra with generators  $e_i, f_i, h_i$  ( $i \in I$ ) and  $d$  subject to the

relations

$$\begin{aligned}
[h_i, h_j] &= 0, & [d, h_i] &= 0, \\
[h_i, e_j] &= a_{ij}e_j, & [d, e_j] &= \delta_{0,j}e_j, \\
[h_i, f_j] &= -a_{ij}f_j, & [d, f_j] &= -\delta_{0,j}f_j, \\
[e_i, f_j] &= \delta_{ij}h_i, \\
(\operatorname{ad}e_i)^{1-a_{ij}}(e_j) &= 0, & (\operatorname{ad}f_i)^{1-a_{ij}}(f_j) &= 0, \quad i \neq j.
\end{aligned}$$

**1.3.** We can define the root system of  $\mathfrak{g}$  in the following way. Extend the root lattice  $\dot{Q}$  of  $\dot{\mathfrak{g}}$  to a lattice  $Q = \dot{Q} \oplus \mathbb{Z}\delta$ , and extend the form  $(\cdot|\cdot)$  to  $Q$  by setting  $(q|\delta) = 0$  for all  $q \in \dot{Q}$  and  $(\delta|\delta) = 0$ . The root system  $\Delta$  of  $\mathfrak{g}$  is given by

$$\Delta = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}, n \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z}, k \neq 0\}.$$

The roots of the form  $\alpha + n\delta$ ,  $\alpha \in \dot{\Delta}, n \in \mathbb{Z}$  are called real roots, and those of the form  $k\delta$ ,  $k \in \mathbb{Z}, k \neq 0$  are called imaginary roots. We let  $\Delta^{re}$  and  $\Delta^{im}$  denote the sets of real and imaginary roots, respectively. The set of positive real roots of  $\mathfrak{g}$  is  $\Delta_+^{re} = \dot{\Delta}_+ \cup \{\alpha + n\delta \mid \alpha \in \dot{\Delta}, n > 0\}$  and the set of positive imaginary roots is  $\Delta_+^{im} = \{k\delta \mid k > 0\}$ . The set of positive roots of  $\mathfrak{g}$  is  $\Delta_+ = \Delta_+^{re} \cup \Delta_+^{im}$ . Let  $Q_+$  be the monoid generated by  $\Delta_+$ . Similarly, on the negative side, we have  $\Delta_- = \Delta_-^{re} \cup \Delta_-^{im}$ , where  $\Delta_-^{re} = \dot{\Delta}_- \cup \{\alpha + n\delta \mid \alpha \in \dot{\Delta}, n < 0\}$  and  $\Delta_-^{im} = \{k\delta \mid k < 0\}$  and let  $Q_-$  be the monoid generated by  $\Delta_-$ . Further, if  $\theta$  denotes the highest positive root of  $\dot{\mathfrak{g}}$  and  $\alpha_0 := \delta - \theta$ , then  $\Pi = \{\alpha_0, \alpha_1, \dots, \alpha_N\}$  is a set of simple roots for  $\mathfrak{g}$ . We extend the weight lattice  $\dot{P}$  of  $\dot{\mathfrak{g}}$  to the weight lattice  $P$  of  $\mathfrak{g}$  defined as  $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \in \mathbb{Z}, i \in I, \lambda(d) \in \mathbb{Z}\}$ . Let  $W$  denote the Weyl group of  $\mathfrak{g}$  generated by the simple reflections  $r_0, r_1, \dots, r_N$  and  $B$  denote the associated braid group with generators  $T_0, T_1, \dots, T_N$ .

**1.4.** Beck [Be1, Be2] has introduced a total ordering of the root system leading to PBW bases for  $\mathfrak{g}$  and its quantum analog,  $U_q(\mathfrak{g})$ . We state the construction here, partially following the more abstract notation developed by Damiani [Da] and Gavarini [Ga]. For a related PBW construction, see [KT].

For any affine algebra  $\mathfrak{g}$ , there exists a map  $\pi : \mathbb{Z} \mapsto I$  such that, if we define

$$\beta_k = \begin{cases} r_{\pi(0)} r_{\pi(-1)} \cdots r_{\pi(k+1)}(\alpha_{\pi(k)}) & \text{for all } k \leq 0 \\ r_{\pi(1)} r_{\pi(2)} \cdots r_{\pi(k-1)}(\alpha_{\pi(k)}) & \text{for all } k \geq 1, \end{cases}$$

then the map  $\pi' : \mathbb{Z} \mapsto \Delta_+^{re}$  given by  $\pi'(k) = \beta_k$  is a bijection. Further, we can choose  $\pi$  so that  $\{\beta_k \mid k \leq 0\} = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \geq 0\}$  and  $\{\beta_k \mid k \geq 1\} = \{-\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n > 0\}$ .

It will be convenient for us to invert Beck's original ordering of the positive roots (cf. [BK, 1.4.1] for the original order and [Ga, §2.1] for this ordering). Thus, we set

$$\beta_0 > \beta_{-1} > \beta_{-2} > \cdots > \delta > 2\delta > \cdots > \beta_2 > \beta_1.$$

Clearly, if we say  $-\alpha < -\beta$  if and only if  $\beta > \alpha$  for all positive roots  $\alpha, \beta$ , we obtain a corresponding ordering on  $\Delta_-$ .

The following elementary observation on the ordering will play a crucial role later. Write  $A < B$  for two sets  $A$  and  $B$  if  $x < y$  for all  $x \in A$  and  $y \in B$ . Then Beck's total ordering of the positive roots can be divided into three sets:

$$\{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \geq 0\} > \{k\delta \mid k > 0\} > \{-\alpha + k\delta \mid \alpha \in \dot{\Delta}_+, k > 0\}.$$

Similarly, for the negative roots, we have,

$$\{-\alpha - n\delta \mid \alpha \in \dot{\Delta}_+, n \geq 0\} < \{-k\delta \mid k > 0\} < \{\alpha - k\delta \mid \alpha \in \dot{\Delta}_+, k > 0\}.$$

Note that the map  $\pi$ , and so the total ordering, is not unique. We assume a suitable  $\pi$  chosen and fixed now throughout the paper. Beck's original approach and proof is constructive, but the existential approach avoids some technicalities we do not need below.

**1.5.** The quantum group, or quantized universal enveloping algebra, of  $\mathfrak{g}$  is the associative algebra  $U_q(\mathfrak{g})$  with 1 over  $\mathbb{C}(q)$  with generators  $E_i, F_i, K_i^{\pm 1}$  ( $i \in I$ ) and  $D^{\pm 1}$  subject to the defining relations

$$\begin{aligned} K_i K_i^{-1} &= K_i^{-1} K_i = D D^{-1} = D^{-1} D = 1, \\ K_i K_j &= K_j K_i, \quad K_i D = D K_i, \\ K_i E_j &= q_i^{a_{ij}} E_j K_i, \quad D E_j = q_0^{\delta_{j,0}} E_j D, \\ K_i F_j &= q_i^{-a_{ij}} F_j K_i, \quad D F_j = q_0^{-\delta_{j,0}} F_j D, \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{k=1}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k &= 0, \quad i \neq j \\ \sum_{k=1}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k &= 0, \quad i \neq j, \end{aligned}$$

where  $q_i = q^{d_i}$  (we can choose  $d_i$  so that  $d_0 = 1$  and  $q_0 = q$ ), and

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[m-n]_q! [n]_q!}, \quad [m]_q! = \prod_{j=1}^m [j]_q, \quad [j]_q = \frac{q^j - q^{-j}}{q - q^{-1}}$$

for all  $i \in I$ ,  $m, n \in \mathbb{Z}$ ,  $m \geq n > 0$ . For any  $\mu \in Q$ , we have  $\mu = \sum_{i \in I} c_i \alpha_i$ , for some integers  $c_i$ . Denote  $K_\mu = \prod_{i \in I} K_i^{c_i}$ . Then  $K_\lambda K_\mu = K_{\lambda+\mu}$  for all  $\lambda, \mu \in Q$ . In particular, we have  $K_{\pm \alpha_i} = K_i^{\pm 1}$ . Let  $U_q^+(\mathfrak{g})$  (resp.  $U_q^-(\mathfrak{g})$ ) be the subalgebra of  $U_q(\mathfrak{g})$  generated by  $E_i$  (resp.  $F_i$ ),  $i \in I$ , and let  $U_q^0(\mathfrak{g})$  denote the subalgebra generated by  $K_i^{\pm 1}$  ( $i \in I$ ) and  $D^{\pm 1}$ .

The action of the braid group generators  $T_i$  on the generators of the quantum

group  $U_q(\mathfrak{g})$  is given by the following.

$$\begin{aligned}
T_i(E_i) &= -F_i K_i, & T_i(F_i) &= -K_i^{-1} E_i, \\
T_i(E_j) &= \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} \frac{1}{[-a_{ij}-r]_{q_i}!} \frac{1}{[-r]_{q_i}!} q_i^{-r} E_i^{-a_{ij}-r} E_j E_i^r, & \text{if } i \neq j, \\
T_i(F_j) &= \sum_{r=0}^{-a_{ij}} (-1)^{r-a_{ij}} \frac{1}{[-r]_{q_i}!} \frac{1}{[-a_{ij}-r]_{q_i}!} q_i^r F_i^r F_j F_i^{-a_{ij}-r}, & \text{if } i \neq j, \\
T_i(K_j) &= K_j K_i^{-a_{ij}}, & T_i(K_j^{-1}) &= K_j^{-1} K_i^{a_{ij}}, \\
T_i(D) &= D K_i^{-\delta_{i,0}}, & T_i(D^{-1}) &= D^{-1} K_i^{\delta_{i,0}}.
\end{aligned}$$

**1.6.** For each  $\beta_k \in \Delta_+^e$ , define the root vector  $E_{\beta_k}$  in  $U_q(\mathfrak{g})$  by

$$E_{\beta_k} = \begin{cases} E_{\pi(0)} & k = 0 \\ T_{\pi(0)}^{-1} T_{\pi(-1)}^{-1} \cdots T_{\pi(k+1)}^{-1} (E_{\pi(k)}) & \text{for all } k < 0 \\ E_{\pi(1)} & k = 1 \\ T_{\pi(1)} T_{\pi(2)} \cdots T_{\pi(k-1)} (E_{\pi(k)}) & \text{for all } k > 1. \end{cases}$$

Each real root space is 1-dimensional, but each imaginary root space is  $N$ -dimensional. Hence, for each positive imaginary root  $k\delta$  ( $k > 0$ ) we define  $N$  imaginary root vectors,  $E_{k\delta}^{(i)}$  ( $i \in I$ ) by

$$\exp \left( (q^i - q^{-i}) \sum_{k=1}^{\infty} E_{k\delta}^{(i)} z^k \right) = 1 + (q^i - q^{-i}) \sum_{k=1}^{\infty} K_i^{-1} [E_i, E_{-\alpha_i+k\delta}] z^k.$$

Then for each  $k$ , the  $E_{k\delta}^{(i)}$  span the  $k\delta$ -root space and commute with each other.

Further, the  $E_{\beta_k}$  ( $k \in \mathbb{Z}$ ) and  $E_{k\delta}^{(i)}$  ( $k > 0$ ) form a basis for  $U_q^+(\mathfrak{g})$ .

Let  $\omega$  denote the standard  $\mathbb{C}$ -linear antiautomorphism of  $U_q(\mathfrak{g})$ , and set  $E_{-\alpha} = \omega(E_{\alpha})$  for all  $\alpha \in \Delta_+$ . Then  $U_q$  has a basis of elements of the form  $E_- H E_+$ , where  $E_{\pm}$  are ordered monomials in the  $E_{\alpha}$ ,  $\alpha \in \Delta_{\pm}$ , and  $H$  is a monomial in  $K_i^{\pm 1}$  and  $D^{\pm 1}$  (which all commute).

Furthermore, this basis is, in Beck's terminology, convex, meaning that, if  $\alpha, \beta \in \Delta_+$  and  $E_{\beta} > E_{\alpha}$ , that is,  $\beta > \alpha$ , then

$$E_{\beta} E_{\alpha} - q^{(\alpha|\beta)} E_{\alpha} E_{\beta} = \sum_{\alpha < \gamma_1 < \cdots < \gamma_r < \beta} c_{\gamma} E_{\gamma_1}^{a_1} \cdots E_{\gamma_r}^{a_r}$$

for some integers  $a_1, \dots, a_r$  and scalars  $c_{\gamma} \in \mathbb{C}[q, q^{-1}]$ ,  $\gamma = (\gamma_1, \dots, \gamma_r)$  [BK, Proposition 1.7c], and similarly for the negative roots.

## 2. Verma-type modules for affine algebras.

We recall here the construction and properties of Verma-type modules for affine algebras. The theory of Verma-type modules has been developed in a number of papers, for example [Co1, Co2, CFM, Fu2, and Fu4]. For convenience, we use [Fu5]

as our standard reference. It contains detailed proofs and references to the original publication of the results summarized below.

The root system  $\Delta$  of  $\mathfrak{g}$  has a natural partition into positive and negative roots,  $\Delta^+$  and  $\Delta^-$ . An arbitrary partition  $\Delta = S \cup -S$  is called *closed* if whenever  $\alpha$  and  $\beta$  are in  $S$  and  $\alpha + \beta$  is a root, then  $\alpha + \beta \in S$ . If  $S$  is a closed partition, then the space  $\mathfrak{g}_S = \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$  is a subalgebra of  $\mathfrak{g}$ , and  $\mathfrak{g}$  has a triangular decomposition  $\mathfrak{g} = \mathfrak{g}_{-S} \oplus \mathfrak{h} \oplus \mathfrak{g}_S$ .

Let  $W$  denote the Weyl group of  $\mathfrak{g}$ . Then it is well-known that, for any finite-dimensional complex simple Lie algebra, all closed partitions are  $W \times \{\pm 1\}$ -equivalent to the standard partition into positive and negative roots. For Kac-Moody algebras this is no longer true. In particular, for affine algebras, there are always a finite number (greater than 1) of inequivalent Weyl-orbits of partitions. These non-standard partitions were first studied and classified by Jakobsen and Kac [JK1, JK2] and Futorny [Fu1, Fu2].

Next we introduce some notation we will need dealing with the root system of  $\mathfrak{g}$  and construct a collection of sets that parametrize closed partitions.

Let  $J \subseteq \dot{I} = \{1, \dots, N\}$ . Let  $\Pi^J = \{\alpha_j \in \Pi \mid j \in J\}$ . Set  $Q^J = \bigoplus_{j \in J} \mathbb{Z}\alpha_j \oplus \mathbb{Z}\delta$ , and  $Q_\pm^J = Q^J \cap Q_\pm$ . For any subset  $\epsilon$  of  $\dot{\Pi}$ , let  $Q_\pm^\epsilon$  denote a semigroup of  $\mathfrak{h}^*$  generated by  $\pm\epsilon$ .

Let  $\dot{\Delta}^J$  be the finite root system generated by the simple roots in  $\Pi^J$ . Then  $\dot{\Delta}^J = \emptyset$  if and only if  $J = \emptyset$ . Set

$$\Delta^J = \{\alpha + n\delta \in \Delta \mid \alpha \in \dot{\Delta}^J, n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z} \setminus \{0\}\},$$

and let  $\Delta_\pm^J = \Delta^J \cap \Delta_\pm$  and  $\Delta_J^\infty = \Delta \setminus \Delta^J$ . Now we let

$$+\Delta_J^\infty = \{\alpha + n\delta \in \Delta \mid \alpha \in \dot{\Delta}_+ \setminus \dot{\Delta}^J, n \in \mathbb{Z}\}.$$

Finally, let  $S_J = \Delta_+^J \cup +\Delta_J^\infty$ .

Note that, for any  $J \subseteq \dot{I}$ , the positive imaginary roots  $k\delta$  ( $k > 0$ ) are in  $\Delta_+^J$  and a simple root  $\alpha_i \in \Delta_+^J$  if and only if  $i \in J$ , while  $\alpha_i \in +\Delta_J^\infty$  if and only if  $i \in \dot{I} \setminus J$ . Hence, regardless of choice of  $J$ , we will always have  $\dot{\Delta}_+ \cup \{k\delta \mid k > 0\} \subseteq S_J$ . Further, we note that  $S_J$  is a closed partition of  $\Delta$  for any  $J \subseteq \dot{I}$ . The two extreme cases are when  $J = \dot{I}$ , in which case  $S_J = \Delta_+$ , and  $J = \emptyset$ , in which case

$$S_\emptyset = \{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \in \mathbb{Z}\} \cup \{k\delta \mid k > 0\}.$$

The sets  $S_J$  parametrize the closed partitions of  $\Delta$ . That is, a closed partition  $S$  is  $W \times \{\pm 1\}$ -equivalent to a unique  $S_J$  for some  $J \subseteq \dot{I}$  [Fu5, Theorem 2.4].

Verma-type modules are representations induced from 1-dimensional representations for nonstandard Borel subalgebras. Equivalence classes of Borel subalgebras correspond to equivalence classes of closed partitions. In order to classify Verma-type modules, we begin by classifying the Borel subalgebras for an affine algebra  $\mathfrak{g}$ . We will always assume the fixed Cartan subalgebra  $\mathfrak{h}$ .

Let  $\sigma$  denote the Chevalley involution of  $\mathfrak{g}$  defined by  $\sigma(e_i) = -f_i, \sigma(f_i) = -e_i, \sigma(h) = -h$  for  $i \in I, h \in \mathfrak{h}$ . Let  $B \subset \mathfrak{g}$  be a Lie subalgebra such that  $\mathfrak{h} \subset B$  and  $B + \sigma(B) = \mathfrak{g}$ . Then we call  $B$  a Borel subalgebra if  $B \cap \sigma(B) = \emptyset$ , and a parabolic subalgebra otherwise. The algebra involution  $\sigma$  induces a linear involutive antiautomorphism  $\sigma$  on the root system  $\Delta$ , whereby  $\sigma(\alpha) = -\alpha$  for all

$\alpha \in \Delta$ . Hence, there is a correspondence between Borel subalgebras of  $\mathfrak{g}$  and closed partitions of  $\Delta$ .

Let  $U(\mathfrak{g}_S)$  (resp.  $U(\mathfrak{g}_{-S})$ ) denote the universal enveloping algebra of  $\mathfrak{g}_S$  (resp.  $\mathfrak{g}_{-S}$ ). Then, by the PBW theorem, the triangular decomposition of  $\mathfrak{g}$  determines a triangular decomposition of  $U(\mathfrak{g})$  as  $U(\mathfrak{g}) = U(\mathfrak{g}_{-S}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}_S)$ .

Let  $\lambda \in \mathfrak{h}^*$ . Then  $\lambda$  extends to a map on  $(U(\mathfrak{h}))^*$ , also denoted by  $\lambda$ . A  $U(\mathfrak{g})$ -module  $V$  is called a weight module if  $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$ , where  $V_\mu = \{v \in V \mid h \cdot v = \mu(h)v \text{ for all } h \in U(\mathfrak{h})\}$ . The non-zero subspaces  $V_\mu$  are called weight spaces. Any submodule of a weight module is a weight module.

Let  $B$  be a Borel subalgebra and  $\lambda : B \rightarrow \mathbb{C}$  a 1-dimensional representation. A weight  $U(\mathfrak{g})$ -module  $V$  is said to be of *highest weight*  $\lambda$  with respect to  $B$  if there is some nonzero vector  $v \in V$  such that  $V = U(\mathfrak{g}) \cdot v$  and  $x \cdot v = \lambda(x)v$  for all  $x \in B$ . We define the induced module

$$M_B(\lambda) = U(\mathfrak{g}) \otimes_{U(B)} \mathbb{C},$$

called the module of *Verma-type* associated with  $B$  and  $\lambda$ . If  $B$  is the standard Borel subalgebra of  $\mathfrak{g}$ , then  $M_B(\lambda)$  is just the usual Verma module with highest weight  $\lambda$ . If  $B$  is not  $W \times \{\pm 1\}$ -conjugate to the standard Borel, then the Verma-type module  $M_B(\lambda)$  contains both finite and infinite-dimensional weight spaces.

Since there is a 1-to-1 correspondence between Borel subalgebras and closed partitions of the root system, and the sets  $S_J$  parametrize the Weyl-equivalence classes of partitions, we can construct a set of representatives of the Borel conjugacy classes and so a canonical collection of Verma-type modules. For  $J \subseteq \dot{I}$ , let  $B^J = \sum_{\beta \in S_J} \mathfrak{g}_\beta \oplus \mathfrak{h}$ . Then  $B^J$  is a Borel subalgebra of  $\mathfrak{g}$ . Denote  $M_{B^J}(\lambda)$  by  $M_J(\lambda)$ . Then the  $M_J(\lambda)$ ,  $J \subseteq \dot{I}$ , parametrize all Verma-type modules. We call  $M_J(\lambda)$  a  $J$ -highest-weight module with  $J$ -highest weight  $\lambda$ . Henceforth, we will consider only the Verma-type modules  $M_J(\lambda)$ .

The *level* of  $M_J(\lambda)$  is  $\lambda(c)$ . At certain points, we need to make the assumption that  $\lambda(c) \neq 0$ . The case of level zero is more complicated and not completely understood. Although some of our techniques do generalize in certain case, we cannot give complete results.

Recalling the two special cases of  $J$ , we have that if  $J = \dot{I}$ , then  $S_J = \Delta_+$  and  $M_{\dot{I}}(\lambda)$  is the ordinary Verma module of highest weight  $\lambda$ . At the other extreme, the modules  $M_\emptyset(\lambda)$  are called *imaginary Verma modules* [Fu3].

We denote by  $\mathfrak{n}_{\pm J}$  the subalgebras  $\mathfrak{g}_{\pm S_J}$ , so we have a triangular decomposition  $\mathfrak{g} = \mathfrak{n}_{-J} \oplus \mathfrak{h} \oplus \mathfrak{n}_J$ .

In the proposition below we collect some basic statements about the structure of the Verma-type modules  $M_J(\lambda)$ . Proofs can be found in [Fu5, Propositions 3.4 and 5.2].

**Proposition 2.1.** *Let  $\lambda \in \mathfrak{h}^*$  and  $J \subseteq \dot{I}$ , and let  $M_J(\lambda)$  be the Verma-type module of  $J$ -highest weight  $\lambda$ . Then  $M_J(\lambda)$  has the following properties.*

- (i) *The module  $M_J(\lambda)$  is a free  $U(\mathfrak{n}_{-J})$ -module of rank 1 generated by the  $S$ -highest weight vector  $1 \otimes 1$  of weight  $\lambda$ .*
- (ii)  *$M_J(\lambda)$  has a unique maximal submodule and hence a unique irreducible quotient, which we denote  $L_J(\lambda)$ .*
- (iii) *Let  $V$  be a  $U(\mathfrak{g})$ -module generated by some  $J$ -highest weight vector  $v$  of weight  $\lambda$ . Then there exists a unique surjective homomorphism  $\phi : M_J(\lambda) \rightarrow V$  such that  $\phi(1 \otimes 1) = v$ .*

- (iv)  $\dim M_J(\lambda)_\mu \neq 0$  if and only if  $\lambda - \mu$  is in the monoid generated by  $S_J$ ,  
 $\dim M_J(\lambda)_\lambda = 1$ ,  
and  $0 < \dim M_J(\lambda)_\mu < \infty$  if and only if  $\lambda - \mu \in Q_+^J$ .

Let  $\mathfrak{g}_J := \sum_{\beta \in \Delta^J} \mathfrak{g}_\beta \oplus \mathfrak{h}$ . We call a subset  $C \subseteq J$  connected if the Coxeter-Dynkin diagram associated to the simple roots  $\alpha_i$ ,  $i \in C$ , is connected. The set  $J$  can then be partitioned into a collection  $\mathcal{C}$  of subsets corresponding to connected components of the Coxeter-Dynkin diagram associated to  $J$ . For  $C \in \mathcal{C}$ , let  $\mathfrak{A}_C$  be the affine subalgebra of  $\mathfrak{g}$  generated by  $e_i, f_i$  ( $i \in C$ ), together with the central element  $c$  and degree derivation  $d$  from  $\mathfrak{g}$ . Let  $\mathfrak{g}^f = \sum_{C \in \mathcal{C}} \mathfrak{A}_C$  and  $\tilde{\mathfrak{g}}^f = \mathfrak{g}^f + \mathfrak{h}$ .

Let  $G := \oplus_{k \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}_{k\delta} \oplus \mathbb{C}c$ . The algebra  $G$  is called a Heisenberg subalgebra of  $\mathfrak{g}$ .  $G$  has a triangular decomposition  $G = G_- \oplus \mathbb{C}c \oplus G_+$ , where  $G_\pm = \oplus_{k > 0} \mathfrak{g}_{\pm k\delta}$ . Set  $\overline{G} = \{g \in G \mid [g, [\mathfrak{g}^f, \mathfrak{g}^f]] = 0\}$ , and set  $\overline{G}_\pm = \overline{G} \cap G_\pm$ . By Theorem 3.3 of [Fu5], we have the following proposition.

**Proposition 2.2.**  $\mathfrak{g}_J = \overline{G}_- \oplus \tilde{\mathfrak{g}}^f \oplus \overline{G}_+$ .

By Proposition 2.1 (iv), we know that  $0 < \dim M_J(\lambda)_\mu < \infty$  if and only if  $\lambda - \mu \in Q_+^J$ . Let us say  $\mu \leq_J \lambda$  if  $\lambda - \mu \in Q_+^J$ . Let  $M_J^f(\lambda) := \oplus_{\mu \leq_J \lambda} M_J(\lambda)_\mu$ . Then  $M_J^f(\lambda)$  is the sum of all the finite-dimensional weight spaces of  $M_J(\lambda)$ . We also set  $L_J^f(\lambda) = \oplus_{\mu \leq_J \lambda} L_J(\lambda)_\mu$ .

**Proposition 2.3** ([Fu5, Proposition 5.3]).  $M_J^f(\lambda) \cong \tilde{M}^f(\lambda) \otimes \overline{M}(\lambda)$ , where  $\tilde{M}^f(\lambda)$  is a Verma module for  $\tilde{\mathfrak{g}}^f$  and  $\overline{M}(\lambda)$  is a Verma module for  $\overline{G}$ .

Next we recall some statements about the submodule structure of  $M_J(\lambda)$ . Up to this point, we have not needed the restriction that  $\lambda(c) \neq 0$ , but we do need it for parts of the next proposition. Let  $\mathfrak{u}_{\pm J} = \oplus_{\beta \in S_J \setminus \Delta^J} \mathfrak{g}_{\pm \beta}$ .

**Proposition 2.4** ([Fu5, Lemma 5.4, Theorem 5.14]). Assume  $\lambda \in \mathfrak{h}^*$ .

- (i) If  $0 \neq v \in M_J(\lambda)$ , then  $U(\mathfrak{g})v \cap M_J^f(\lambda) \neq 0$ .
- (ii) If  $N$  is a  $\mathfrak{g}$ -submodule of  $M_J(\lambda)$  and  $\lambda(c) \neq 0$ , then

$$N \cong U(\mathfrak{u}_{-J}) \otimes_{\mathbb{C}} (N \cap M_J^f(\lambda))$$

as vector spaces.

- (iii) If  $\lambda(c) \neq 0$ , then  $L_J(\lambda) \cong U(\mathfrak{u}_{-J}) \otimes_{\mathbb{C}} L_J^f(\lambda)$  as vector spaces.

We have  $\lambda \in \mathfrak{h}^*$ . For  $C \in \mathcal{C}$ , let  $\lambda_C = \lambda|_{(\mathfrak{h} \cap \mathfrak{A}_C)^*}$ , and let  $M^C(\lambda)$  denote the Verma module over  $\mathfrak{A}_C$  with highest weight  $\lambda_C$ . Since  $\tilde{M}^f(\lambda)$  is isomorphic to a tensor product of  $M^C(\lambda)$ , we obtain from Proposition 2.4 the following irreducibility criterion.

**Corollary 2.5.** The Verma-type module  $M_J(\lambda)$  is irreducible if and only if  $\lambda(c) \neq 0$  and  $M^C(\lambda)$  is an irreducible  $\mathfrak{A}_C$ -module for every  $C \in \mathcal{C}$ .

In particular, for the special case of imaginary Verma modules, we have the following simple criterion.

**Corollary 2.6** ([Fu5, Proposition 5.8]). The imaginary Verma module  $M_\emptyset(\lambda)$  is irreducible if and only if  $\lambda(c) \neq 0$ .

### 3. Verma-type modules for quantum affine algebras.

Next we must construct the quantum modules that we will show are deformations of the classical Verma-type modules. Let  $J \subseteq \dot{I}$ . Let  $\mathfrak{n}_{\pm J}^q$  be the subalgebra of  $U_q(\mathfrak{g})$  generated by  $\{E_\beta \mid \beta \in \pm S_J\}$ , and let  $B_q$  denote the subalgebra of  $U_q(\mathfrak{g})$  generated by  $\{E_\beta \mid \beta \in S_J\} \cup U_q^0$ .

A  $U_q(\mathfrak{g})$ -module  $V^q$  is called a quantum weight module if  $V^q = \bigoplus_{\mu \in P} V_\mu^q$ , where

$$V_\mu^q = \{v \in V \mid K_i^{\pm 1} \cdot v = q_i^{\pm \mu(h_i)} v, D^{\pm 1} \cdot v = q_0^{\pm \mu(d)} v\}.$$

Any submodule of a quantum weight module is a weight module. A  $U_q(\mathfrak{g})$ -module  $V^q$  is called a  $J$ -highest weight module with highest weight  $\lambda \in P$  if there is a non-zero vector  $v \in V^q$  such that:

- (i)  $u^+ \cdot v = 0$  for all  $u^+ \in \mathfrak{n}_J^q \setminus \mathbb{C}(q)^*$ ;
- (ii) for each  $i \in I$ ,  $K_i^{\pm 1} \cdot v = q_i^{\pm \lambda(h_i)} v$ ,  $D^{\pm 1} \cdot v = q_0^{\pm \lambda(d)} v$ ;
- (iii)  $V^q = U_q(\mathfrak{g}) \cdot v$ .

Note that, in the absence of a general quantum PBW theorem for non-standard partitions, we cannot immediately claim that a  $J$ -highest weight module  $V^q$  is generated by  $\mathfrak{n}_{-J}^q$ . This is in contrast to the classical case, and the reason behind Theorem 3.5 below.

We define a  $U_q(\mathfrak{g})$ -module as follows. Let  $\mathbb{C}(q) \cdot v$  be a 1-dimensional vector space. Let  $\lambda \in P$ , and set  $E_\beta \cdot v = 0$  for all  $\beta \in S_J$ ,  $K_i^{\pm 1} \cdot v = q_i^{\pm \lambda(h_i)} v$  ( $i \in I$ ) and  $D^{\pm 1} \cdot v = q_0^{\pm \lambda(d)} v$ . Now define  $M_J^q(\lambda) = U_q(\mathfrak{g}) \otimes_{B_q} \mathbb{C}(q)v$ . Then  $M_J^q(\lambda)$  is a  $J$ -highest weight  $U_q$ -module called the *quantum Verma-type module* with  $J$ -highest weight  $\lambda$ .

The following contains the basic properties of Verma-type modules.

**Proposition 3.1.** *Let  $\lambda \in P$  and  $J \subseteq \dot{I}$ , and let  $M_J^q(\lambda)$  be the quantum Verma-type module of  $J$ -highest weight  $\lambda$ . Then  $M_J^q(\lambda)$  has the following properties.*

- (i) *The module  $M_J^q(\lambda)$  is a free  $\mathfrak{n}_{-J}^q$ -module of rank 1 generated by the  $J$ -highest weight vector  $v_\lambda$  of weight  $\lambda$ .*
- (ii)  *$M_J^q(\lambda)$  has a unique maximal submodule and hence a unique irreducible quotient, which we denote  $L_J^q(\lambda)$ .*
- (iii) *Let  $V$  be a  $U_q(\mathfrak{g})$ -module generated by some  $J$ -highest weight vector  $v$  of weight  $\lambda$ . Then there exists a unique surjective homomorphism  $\phi : M_J^q(\lambda) \mapsto V$  such that  $\phi(v_\lambda) = v$ .*

We also want to show that the module  $M_J^q(\lambda)$  is spanned by the “right” set of vectors. In order to do this, we must appeal to Beck’s PBW basis of  $U_q(\mathfrak{g})$  and to the very useful grading by degree introduced by Beck and Kac [BK, §1.8], which we reproduce here as our notation differs slightly from theirs.

Beck [Be2] has shown that  $U_q(\mathfrak{g})$  has a basis comprising elements of the form  $N_{(a_\beta)} K M_{(a'_\beta)}$ , where the  $M_{(a_\beta)}$  are ordered monomials in  $E_\beta^{a_\beta}$ ,  $\beta \in \Delta_+$ ,  $a_\beta \in \mathbb{Z}_+$ ,  $N_{(a_\beta)} = \omega(M_{(a_\beta)})$ , and  $K$  is an ordered monomial in  $K_i^{\pm 1}$  and  $D^{\pm 1}$ . The notation  $(a_\beta)$  indicates the sequence of powers  $a_\beta$  as  $\beta$  runs over  $\Delta_+$ . Of course, almost all terms of the sequence are zero.

In [BK, §1.8], Beck and Kac define the *total height* of such a basis element by

$$d_0(N_{(a_\beta)} K M_{(a'_\beta)}) = \sum_{\beta \in \Delta_+} (a_\beta + a'_\beta) \text{ht} \beta,$$

where  $\text{ht}\beta$  is the usual height of a root. Next, they set the *total degree* of a basis element to be

$$d(N_{(a_\beta)}KM_{(a'_\beta)}) = (d_0(N_{(a_\beta)}KM_{(a'_\beta)}), (a_\beta), (a'_\beta)) \in \mathbb{Z}_+^{2\Delta_++1}.$$

Considering  $\mathbb{Z}_+^{2\Delta_++1}$  as a totally ordered semigroup with the usual lexicographical ordering, Beck and Kac introduce a filtration of  $U_q(\mathfrak{g})$  by defining  $U_s$ , for any  $s \in \mathbb{Z}_+^{2\Delta_++1}$ , to be the span of the basis monomials  $N_{(a_\beta)}KM_{(a'_\beta)}$  with degree  $d(N_{(a_\beta)}KM_{(a'_\beta)}) \leq s$ . Finally, they obtain the following proposition.

**Proposition 3.2** ([BK, Proposition 1.8]). *The associated graded algebra  $\text{Gr}U_q(\mathfrak{g})$  of the  $\mathbb{Z}_+^{2\Delta_++1}$ -filtered algebra  $U_q(\mathfrak{g})$  is the algebra over  $\mathbb{C}(q)$  generated by  $E_\alpha$ ,  $\alpha \in \Delta$ , counting multiplicities,  $K_i^{\pm 1}$  ( $i \in I$ ), and  $D^\pm$  subject to the relations*

$$\begin{aligned} K_i K_i^{-1} &= K_i^{-1} K_i = DD^{-1} = D^{-1}D = 1, \\ K_i K_j &= K_j K_i, \quad K_i D = D K_i, \\ K_i E_\alpha &= q^{(\alpha_i|\alpha)} E_\alpha K_i, \quad DE_\alpha = q^n E_\alpha D, \text{ for } \alpha = \gamma + n\delta, \gamma \in \dot{\Delta}. \\ E_\alpha E_{-\beta} &= E_{-\beta} E_\alpha \text{ if } \alpha, \beta \in \Delta_+, \\ E_\alpha E_\beta &= q^{(\alpha|\beta)} E_\beta E_\alpha, \quad E_{-\alpha} E_{-\beta} = q^{(\alpha|\beta)} E_{-\beta} E_{-\alpha}, \text{ if } \alpha, \beta \in \Delta_+ \text{ and } \beta < \alpha. \end{aligned}$$

Next we need the following technical result, which we state in an abstract manner. Let  $\mathcal{I}$  be a totally ordered set without infinite decreasing chains (i.e.,  $\mathcal{I}$  has a minimal element with respect to the ordering). Let  $A$  be an associative algebra over a field  $K$  with generators  $\{a_i \mid i \in \mathcal{I}\}$ . Let  $O = \{(n_i \mid i \in \mathcal{I}, n_i \in \mathbb{Z}_+, n_i = 0 \text{ for all but finite number of indices})\}$ . The set  $O$  has a total lexicographical order such that  $(n_i \mid i \in \mathcal{I}) > (m_i \mid i \in \mathcal{I})$  if and only if there exists some  $j \in \mathcal{I}$  such that  $n_j > m_j$  and  $n_k = m_k$  for all  $k > j$ .

Suppose that the algebra  $A$  has a basis  $B = \{v = a_{i_1}^{k_1} \dots a_{i_s}^{k_s} \mid i_1 > \dots > i_s\}$ . Then the orderings of  $\mathcal{I}$  and  $O$  impose an ordering on  $B$ . For any word  $v = a_{j_1} \dots a_{j_k} \in A$ , denote by  $\bar{v}$  the unique element in  $B$  such that  $v = \lambda \bar{v} + (\text{terms lower in the ordering})$ , for some  $\lambda \in K$ . If  $\bar{v} = a_{i_1}^{k_{i_1}} \dots a_{i_s}^{k_{i_s}}$  as an element of  $B$ , set  $|v| = (k_{i_1}, \dots, k_{i_s}) \in O$ .

**Proposition 3.3.** *Suppose that for all  $i$  and  $j$  in  $\mathcal{I}$  we have: (\*)  $a_i a_j = \xi_{ij} a_j a_i + \sum_{v \in B} \xi_v v$ , where  $\xi_{ij} \neq 0$  and  $|a_i a_j| > |v|$  for all  $v$  such that  $\xi_v \neq 0$ . Then  $|vw| = |v| + |w|$  for all  $v, w \in B$ .*

*Proof.* We proceed by induction on  $|w|$  and, for fixed  $|w|$ , by induction on  $|v|$ . By our assumption the set  $\mathcal{I}$  has a minimal element  $i_0$ . If  $|w| = \epsilon_{i_0} = (n_i \mid n_i = 0, i \neq i_0, n_{i_0} = 1)$  then  $w$  is the minimal element in  $O$ . Hence, for any  $v \in B$ , we have  $\bar{v}w = vw$  and so  $|vw| = |\bar{v}w| = |v| + |w|$ .

Let  $w = a_i$  and  $v = v_1 a_j$ . If  $j \geq i$ , then  $\bar{v}w = vw$  and  $|vw| = |v| + |w|$ . If  $j < i$ , then by (\*),  $vw = v_1 a_j a_i = \xi_{ij} v_1 a_i a_j + \sum_{u \in B} \xi_u v_1 u$ . We will prove that  $|v_1 u| < |v_1 a_i a_j|$ . By induction on  $|v|$  we have  $|v_1 a_i| = |v_1| + |a_i|$ , while by induction on  $|w|$  we get  $|v_1 a_i a_j| = |v_1 a_i| + |a_j| = |v_1| + |a_i| + |a_j|$ . Let  $\xi_u \neq 0$  and  $u = a_k u_1$ . If  $k < i$  then by induction on  $|w|$  we have  $|v_1 u| = |v_1| + |u|$ , since  $|u| < |a_i|$ . If  $k = i$  then  $u_1 = a_s u_2$  and  $s < j < i$  since  $|u| = |a_i a_s u_2| < |a_i a_j|$ .

Then  $|v_1 u| = |v_1 a_i a_s u_2| = |v_1 a_i| + |a_s u_2|$  and, as  $|a_i| > |a_s u_2|$ , we also have  $|v_1 a_i| + |a_s u_2| = |v_1| + |a_i| + |a_s u_2| < |v_1| + |a_i| + |a_j|$ , since  $s < j$ . Hence we have  $|v_1 u| < |v_1 a_i a_j|$  if  $\xi_u \neq 0$  and  $|vw| = |v_1 a_i a_j| = |v_1| + |a_i| + |a_j| = |v| + |a_i|$ .

Now assume that  $w = w_1 a_i$  with  $|w| > |w_1| > |a_i|$ . By induction,  $|vw| = |vw_1 a_i| = |vw_1| + |a_i| = |v| + |w_1| + |a_i| = |v| + |w|$ . Hence, the proposition is proved.  $\square$

Let  $u$  be an arbitrary element of  $U_q(\mathfrak{g})$ . We may write  $u$  uniquely as a sum of basis monomials and define the total degree of  $u$  to be the largest total degree of these basis elements. We make the following observation.

**Proposition 3.4.** *Let  $u \in U_q(\mathfrak{g})$  be an arbitrary element and  $u', u''$  two basis monomials with  $d(u') < d(u'')$ . Then  $d(uu') < d(uu'')$  and  $d(u'u) < d(u''u)$ .*

*Proof.* The result follows from Propositions 3.2 and 3.3.  $\square$

Next, we give the main result of this section. First, however, we introduce some notation to clarify the argument. We will need these definitions again in Section 6.

Consider the following subsets of  $\Delta$ :

$$\begin{aligned} A_1 &= \{\alpha + n\delta \mid \alpha \in \dot{\Delta}_+, n \geq 0\}, \\ A_2 &= \{k\delta \mid k > 0\}, \\ A_3 &= \{-\alpha + k\delta \mid \alpha \in \dot{\Delta}_+, k > 0\}, \\ B_1 &= \{-\alpha - n\delta \mid \alpha \in \dot{\Delta}_+, n \geq 0\}, \\ B_2 &= \{-k\delta \mid k > 0\}, \\ B_3 &= \{\alpha - k\delta \mid \alpha \in \dot{\Delta}_+, k > 0\}. \end{aligned}$$

Then  $\Delta_+ = A_1 \cup A_2 \cup A_3$  and  $\Delta_- = B_1 \cup B_2 \cup B_3$ . Note that, in our ordering of the root system, we have

$$B_1 < B_2 < B_3 < A_3 < A_2 < A_1.$$

Now we must split these subsets into those associated with finite and infinite-dimensional subspaces. For  $i = 1, 2, 3$ , let  $A_i^{fin} = A_i \cap \Delta^J$ ,  $A_i^\infty = A_i \cap \Delta_J^\infty$ ,  $B_i^{fin} = B_i \cap \Delta^J$  and  $B_i^\infty = B_i \cap \Delta_J^\infty$ . Now, let  $X_i$  denote an ordered monomial in elements  $E_\beta$ ,  $\beta \in A_i$ , and  $Y_i$  denote an ordered monomial of elements  $E_\beta$ ,  $\beta \in B_i$ . Further, let  $X_i^{fin}$  (resp.  $X_i^\infty$ ) denote an ordered monomial in  $E_\beta$ ,  $\beta \in A_i^{fin}$  (resp.  $\beta \in A_i^\infty$ ), and let  $Y_i^{fin}$  (resp.  $Y_i^\infty$ ) denote an ordered monomial in  $E_\beta$ ,  $\beta \in B_i^{fin}$  (resp.  $\beta \in B_i^\infty$ ). We note that the sets  $A_2^\infty$  and  $B_2^\infty$  are actually empty.

**Theorem 3.5.** *As a vector space over  $\mathbb{C}(q)$ ,  $M_J^q(\lambda)$  is isomorphic to the space spanned by the ordered monomials*

$$E_{-\beta-n\delta} \dots E_{-\beta+k\delta} \dots E_{-\alpha-n\delta} \dots E_{-k\delta} \dots E_{\alpha-k\delta},$$

for  $\alpha \in \dot{\Delta}_+ \cap \Delta^J$ ,  $\beta \in \dot{\Delta}_+ \cap \Delta_J^\infty$ ,  $n \geq 0$ ,  $k > 0$ .

*Proof.* Utilizing Beck's PBW basis, we know any element  $u \in U_q(\mathfrak{g})$  can be written in the form

$$u = \sum Y_1 Y_2 Y_3 Z X_3 X_2 X_1,$$

where  $Z \in U_q^0(\mathfrak{g})$ .

Let  $v_\lambda$  be the canonical generator of  $M_J^q(\lambda)$ . Suppose  $w \in M_J^q(\lambda)$ . Then, since  $M_J^q(\lambda) = U_q(\mathfrak{g}) \cdot v_\lambda$ , we have  $w = u \cdot v_\lambda$  for some  $u \in U_q(\mathfrak{g})$ . In view of the discussion above, we may write  $w = \sum Y_1 Y_2 Y_3 Z X_3 X_2 X_1 \cdot v_\lambda$ , for suitable monomials  $X_i, Y_i, Z$ .

As observed in Section 2, all roots of the form  $\alpha + n\delta$  and  $k\delta$  ( $\alpha \in \dot{\Delta}_+, n \geq 0, k > 0$ ) are in  $S_J$  for all subsets  $J \subseteq \dot{I}$ . Hence, monomials of the form  $X_1$  and  $X_2$  all act as 0 on  $v_\lambda$ . Further,  $Z$  commutes with  $X_3$  up to a scalar in  $\mathbb{C}(q)$ . Hence, we can write  $w = \sum Y_1 Y_2 Y_3 X_3 \cdot v_\lambda$ . The theorem asserts that  $M_J^q(\lambda)$  is spanned by monomials of the form  $Y_1^\infty X_3^\infty Y_1^{fin} Y_2 Y_3^{fin} \cdot v_\lambda$ . We proceed to rearrange the monomials in steps, arguing by induction on the total degree. The base of the induction is trivial as, if  $d_0 = 1$ , there is only one simple root involved and nothing to do. Now suppose the theorem is true for all monomials up to total degree  $d$ . We consider monomials of next highest degree.

First, we show that we can reduce to considering expressions of the form  $Y_1 Y_2 Y_3 X_3^\infty \cdot v_\lambda$ . To do this, we have to show that any monomial of the form  $X_3^{fin}$  acts as 0 on  $v_\lambda$ , and that we can move all terms in  $X_3^{fin}$  to the right of any terms from  $X_3^\infty$ .

Since  $A_3 \subset \Delta^+$ , if a root  $-\alpha + k\delta$  is in  $A_3^{fin} = A_3 \cap \Delta^J$ , then  $-\alpha + k\delta \in \Delta_+^J$  and so  $-\alpha + k\delta \in S_J$ . Thus,  $E_{-\alpha+k\delta} \cdot v_\lambda = 0$ , and we have  $X_3^{fin} \cdot v_\lambda = 0$ .

Suppose a factor of the form  $E_{-\alpha+k\delta} E_{-\beta+l\delta}$  with  $-\alpha+k\delta \in A_3^{fin}$  and  $-\beta+l\delta \in A_3^\infty$  occurs in a monomial  $X_3$ . Since  $E_{-\alpha+k\delta}$  and  $E_{-\beta+l\delta}$  are both positive root vectors, we can use the convexity properties of Beck's basis and the grading of the algebra to see that

$$E_{-\alpha+k\delta} E_{-\beta+l\delta} = q^{\pm(\alpha|\beta)} E_{-\beta+l\delta} E_{-\alpha+k\delta} + f E_{-(\alpha+\beta)+(k+l)\delta} + \sum f_\gamma E_\gamma,$$

where  $\deg E_\gamma < \deg E_{-(\alpha+\beta)+(k+l)\delta}$  and  $f$  and  $f_\gamma$  are scalars in  $\mathbb{C}[q, q^{-1}]$ . The sign of  $q^{\pm(\alpha|\beta)}$  depends on whether  $-\alpha + k\delta > -\beta + l\delta$  or not.

The monomial  $X_3'$  obtained from  $X_3$  by replacing the factor  $E_{-\alpha+k\delta} E_{-\beta+l\delta}$  by  $E_{-\beta+l\delta} E_{-\alpha+k\delta}$  has the  $X_3^{fin}$  term moved to the right. If  $-\alpha + k\delta \in A_3^{fin}$  and  $-\beta + l\delta \in A_3^\infty$ , then  $-(\alpha + \beta) + (k + l)\delta \in A_3^\infty$ . Hence, the monomial  $X_3''$  with  $E_{-\alpha+k\delta} E_{-\beta+l\delta}$  replaced by  $E_{-(\alpha+\beta)+(k+l)\delta}$  has the same total degree but one fewer term in  $X_3^{fin}$ . We may repeat the process as necessary. Finally, the monomials  $X_3'$  with  $E_{-\alpha+k\delta} E_{-\beta+l\delta}$  replaced by  $E_\gamma$  are of lower total degree by Proposition 3.4, and can be rearranged by the inductive hypothesis. Therefore, we need only consider the action of monomials of the form  $Y_1 Y_2 Y_3 X_3^\infty$ .

Secondly, we must determine how to commute monomials of the forms  $Y_3$  and  $X_3^\infty$ . Let

$$\begin{aligned} X_3^\infty &= E_{-\alpha_1+k_1\delta} \cdots E_{-\alpha_r+k_r\delta}, \text{ and} \\ Y_3 &= E_{\beta_1-m_1\delta} \cdots E_{\beta_s-m_s\delta}, \end{aligned}$$

for suitable roots  $-\alpha_i + k_i\delta \in A_3^\infty$  ( $i = 1, \dots, r$ ) and  $\beta_j - m_j\delta \in B_3$  ( $j = 1, \dots, s$ ). Then we can write

$$\begin{aligned} Y_3 X_3^\infty &= E_{\beta_1-m_1\delta} \cdots E_{\beta_s-m_s\delta} E_{-\alpha_1+k_1\delta} \cdots E_{-\alpha_r+k_r\delta} \\ &= E_{\beta_1-m_1\delta} \cdots E_{\beta_{s-1}-m_{s-1}\delta} E_{-\alpha_1+k_1\delta} E_{\beta_s-m_s\delta} E_{-\alpha_2+k_2\delta} \cdots E_{-\alpha_r+k_r\delta} \\ &\quad + (\text{terms of lower total degree}) \end{aligned}$$

by using the grading of Proposition 3.2. Repeating this process we get that

$$Y_3 X_3^\infty \cdot v_\lambda = X_3^\infty Y_3 \cdot v_\lambda + (\text{terms of lower degree}) \cdot v_\lambda.$$

By induction on the total degree, we may order the terms of lower degree (as they act on  $v_\lambda$ ), and we have reduced the monomials we are concerned about to those of the form  $Y_1 Y_2 X_3^\infty Y_3$ .

For the next step in the reduction process, we have to show that any monomial of the form  $Y_3^\infty$  acts as 0 on  $v_\lambda$  and that in  $Y_3$  we can move all terms in  $Y_3^\infty$  to the right of any terms in  $Y_3^{fin}$ .

Suppose  $Y_3^\infty$  contains a root vector  $E_{\beta-k\delta}$ . Then  $\beta - k\delta \in A_3 \cap \Delta_J^\infty$ . Since  $\beta - k\delta \in \Delta_J^\infty$ , we have  $\beta - k\delta \notin \Delta^J$  and so  $\beta \notin \dot{\Delta}^J$ . But since  $\beta - k\delta \in A_3$ ,  $\beta \in \dot{\Delta}_+$  and we must have  $\beta \in \dot{\Delta}_+ \setminus \dot{\Delta}^J$ . Thus,  $\beta - k\delta \in {}_+\Delta_J^\infty \subset S_J$  and  $E_{\beta-k\delta} \cdot v_\lambda = 0$ . Hence,  $Y_3^\infty \cdot v_\lambda = 0$ .

Now suppose a factor of the form  $E_{\beta-k\delta} E_{\alpha-l\delta}$ , with  $\beta - k\delta \in B_3^\infty$  and  $\alpha - l\delta \in B_3^{fin}$  occurs in a monomial  $Y_3$ . Then both  $E_{\beta-k\delta}$  and  $E_{\alpha-l\delta}$  are negative roots and we can proceed with a similar argument to that used for showing we can move factors in  $X_3^{fin}$  to the right of those in  $X_3^\infty$ , subject to our inductive hypothesis. Note that in this case, if  $\beta - k\delta \in B_3^\infty$  and  $\alpha - l\delta \in B_3^{fin}$ , then  $(\alpha + \beta) - (k + l)\delta \in B_3^\infty$  and the  $Y_3^\infty$  factor absorbs the  $Y_3^{fin}$  factor.

We are now reduced to rearranging monomials of the form  $Y_1 Y_2 X_3^\infty Y_3^{fin}$ . The fourth step is to show that we can move monomials of the form  $Y_2$  to the right of those of the form  $X_3^\infty$ .

Consider a monomial  $m$  containing a factor  $E_{-k\delta} E_{-\alpha+l\delta}$ ,  $k > 0$ ,  $-\alpha + l\delta \in A_3^\infty$ . Then  $E_{-k\delta}$  is a negative root vector and  $E_{-\alpha+l\delta}$  is a positive root vector and in the graded algebra, positive and negative root vectors commute. Thus, if  $m'$  is the monomial obtained from  $m$  by replacing the factor  $E_{-k\delta} E_{-\alpha+l\delta}$  by  $E_{-\alpha+l\delta} E_{-k\delta}$ , then we have

$$m = m' + (\text{terms of lower total degree}).$$

The new terms of lower total degree may be disarranged, but, by the inductive hypothesis, can be given the desired ordering.

The next step is to show that we can write monomials  $Y_1$  in the form  $Y_1^\infty Y_1^{fin}$ . All root vectors in  $Y_1$  are negative and the argument is similar to that used in the case of  $X_3$  and  $Y_1$ .

The final step is to show that, up to terms of lower degree, we can move monomials of the form  $Y_1^{fin}$  to the right of those of the form  $X_3^\infty$ . The argument here is similar to that of the fourth step of the argument when we moved monomials of the form  $Y_2$  to the right of  $X_3^\infty$ .

It is important to note in this argument that, first, we are rearranging monomials as they act on  $v_\lambda$ . We do not claim that these monomials are equivalent in the algebra  $U_q(\mathfrak{g})$ . That is, we are not claiming a PBW result. Secondly, the ‘‘commutations’’ we have considered take place with respect to the grading. These monomials do not commute in the algebra, or even directly with respect to their action on  $v_\lambda$ . At each stage, we may acquire more monomials, but these additional terms can be reordered by the inductive hypothesis.  $\square$

#### 4. $\mathbb{A}$ -forms of Verma-type modules.

In the previous section, we constructed quantum Verma-type modules. Now we show that these quantum Verma-type modules are quantum deformations of Verma-type modules defined over the affine algebra. To do this, we need to show

that the weight-space structure of a given module  $M_J^q(\lambda)$  is the same as that of its classical counterpart  $M_J(\lambda)$  for any  $\lambda \in P$  and  $J \subseteq \dot{I}$ . The first step is to construct an intermediate module, called an  $\mathbb{A}$ -form.

Following [Lu], for each  $i \in I$ ,  $s \in \mathbb{Z}$  and  $n \in \mathbb{Z}_+$ , we define the *Lusztig elements* in  $U_q(\mathfrak{g})$ :

$$\begin{aligned} \begin{bmatrix} K_i & s \\ n \end{bmatrix} &= \prod_{r=1}^n \frac{K_i q_i^{s-r+1} - K_i^{-1} q_i^{-(s-r+1)}}{q_i^r - q_i^{-r}}, \\ \begin{bmatrix} D & s \\ n \end{bmatrix} &= \prod_{r=1}^n \frac{D q_0^{s-r+1} - D^{-1} q_0^{-(s-r+1)}}{q_0^r - q_0^{-r}}. \end{aligned}$$

Let  $\mathbb{A} = \mathbb{C}[q, q^{-1}, \frac{1}{[n]_{q_i}}, i \in I, n > 0]$ . Define the  $\mathbb{A}$ -form,  $U_{\mathbb{A}}(\mathfrak{g})$ , of  $U_q(\mathfrak{g})$  to be the  $\mathbb{A}$ -subalgebra of  $U_q(\mathfrak{g})$  with 1 generated by the elements  $E_i, F_i, K_i^{\pm 1}, \begin{bmatrix} K_i & 0 \\ 1 \end{bmatrix}, i \in I, D^{\pm 1}, \begin{bmatrix} D & 0 \\ 1 \end{bmatrix}$ . Let  $U_{\mathbb{A}}^+$  (resp.  $U_{\mathbb{A}}^-$ ) denote the subalgebra of  $U_{\mathbb{A}}$  generated by the  $E_i$ , (resp.  $F_i$ ),  $i \in I$ , and let  $U_{\mathbb{A}}^0$  denote the subalgebra of  $U_{\mathbb{A}}$  generated by the elements  $K_i^{\pm 1}, \begin{bmatrix} K_i & 0 \\ 1 \end{bmatrix}, i \in I, D^{\pm 1}, \begin{bmatrix} D & 0 \\ 1 \end{bmatrix}$ .

For any  $i \in I$ ,  $s \in \mathbb{Z}$  and  $n \in \mathbb{Z}_+$ , we have the following identity

$$\begin{bmatrix} K_i & s \\ n \end{bmatrix} = \prod_{i=1}^n \frac{1}{[r]_{q_i}} \left( \begin{bmatrix} K_i & 0 \\ 1 \end{bmatrix} + [s-r+1]_{q_i} K_i^{-1} \right) \quad (\text{cf. [BKMe, eq. 3.8]}).$$

Hence, all  $\begin{bmatrix} K_i & s \\ n \end{bmatrix}$  are in  $U_{\mathbb{A}}$ . Similarly, all  $\begin{bmatrix} D & s \\ n \end{bmatrix}$  are also in  $U_{\mathbb{A}}$ .

**Proposition 4.1.** *The following commutation relations hold between the generators of  $U_{\mathbb{A}}$ . For  $i, j \in I$ ,  $s \in \mathbb{Z}$ ,  $n \in \mathbb{Z}_+$ ,*

$$\begin{aligned} E_i \begin{bmatrix} K_j & s \\ n \end{bmatrix} &= \begin{bmatrix} K_j & s - a_{ij} \\ n \end{bmatrix} E_i, \\ \begin{bmatrix} K_j & s \\ n \end{bmatrix} F_i &= F_i \begin{bmatrix} K_j & s - a_{ij} \\ n \end{bmatrix}, \\ E_i \begin{bmatrix} D & s \\ n \end{bmatrix} &= \begin{bmatrix} D & s - \delta_{i,0} \\ n \end{bmatrix} E_i, \\ \begin{bmatrix} D & s \\ n \end{bmatrix} F_i &= F_i \begin{bmatrix} D & s - \delta_{i,0} \\ n \end{bmatrix}, \\ E_i F_j &= F_j E_i, \quad \text{for } i \neq j, \\ E_i F_i^n &= F_i^n E_i + F_i^{n-1} \sum_{r=0}^{n-1} \begin{bmatrix} K_i & -2r \\ 1 \end{bmatrix}. \end{aligned}$$

*Proof.* The first five equalities follow from the defining relations of  $U_q(\mathfrak{g})$  and the definition of the Lusztig elements, while the last equality is proved by induction.  $\square$

An immediate consequence of Proposition 4.1 is that  $U_{\mathbb{A}}$  inherits the standard triangular decomposition of  $U_q(\mathfrak{g})$ . In particular, any element  $u$  of  $U_{\mathbb{A}}$  can be

written as a sum of monomials of the form  $u^- u^0 u^+$  where  $u^\pm \in U_{\mathbb{A}}^\pm$  and  $u^0 \in U_{\mathbb{A}}^0$ . In fact, we can say rather more. For each positive real root  $\beta$ , the root vector  $E_\beta$  in Beck's basis is defined via the action of the braid group on the generators  $E_i$ . But the coefficients of this action are all in the ring  $\mathbb{A}$ . Consequently, the real root vectors are in  $U_{\mathbb{A}}$ . Next, consider the definition of the positive imaginary root vectors  $E_{k\delta}^{(i)}$ ,  $i \in I$ ,  $k > 0$ . These are given in terms of an exponential generating function containing commutators of the form  $[E_i, E_{-\alpha_i+k\delta}]$ , and these will also be in  $U_{\mathbb{A}}$  since all the  $E_i$  and  $E_{-\alpha_i+k\delta}$  are. The  $\mathbb{C}(q)$  coefficients of the generating function are all in  $\mathbb{A}$ , and so the imaginary root vectors are all in  $U_{\mathbb{A}}$ . Thus,  $U_{\mathbb{A}}$  inherits from  $U_q(\mathfrak{g})$  a basis of monomials of the form  $N_{(a_\beta)} K M_{(a'_\beta)}$ , where  $M_{(a'_\beta)}$  and  $N_{(a_\beta)}$  are as before, and  $K$  is now an (ordered) monomial in the generators  $K_i^\pm$ ,  $\begin{bmatrix} K_i & 0 \\ & 1 \end{bmatrix}$ ,  $i \in I$ ,  $D^\pm$  and  $\begin{bmatrix} D & 0 \\ & 1 \end{bmatrix}$  of  $U_{\mathbb{A}}^0$ .

Let  $\lambda \in P$ ,  $J \subseteq \dot{I}$  and let  $M_J^q(\lambda)$  be the Verma-type module over  $U_q(\mathfrak{g})$  with  $J$ -highest weight  $\lambda$  and highest weight vector  $v_\lambda$ . The  $\mathbb{A}$ -form of  $M_J^q(\lambda)$ ,  $M_J^{\mathbb{A}}(\lambda)$ , is defined to be the  $U_{\mathbb{A}}$  submodule of  $M_J^q(\lambda)$  generated by  $v_\lambda$ . That is, we set

$$M_J^{\mathbb{A}}(\lambda) = U_{\mathbb{A}} \cdot v_\lambda.$$

**Proposition 4.2.** *As a vector space over  $\mathbb{A}$ ,  $M_J^{\mathbb{A}}(\lambda)$  is isomorphic to the space spanned by the ordered monomials*

$$E_{-\beta-n\delta} \dots E_{-\beta+k\delta} \dots E_{-\alpha-n\delta} \dots E_{-k\delta} \dots E_{\alpha-k\delta},$$

for  $\alpha \in \dot{\Delta}_+ \cap \Delta^J$ ,  $\beta \in \dot{\Delta}_+ \cap \Delta_J^\infty$ ,  $n \geq 0$ ,  $k > 0$ .

*Proof.* As in the proof of Theorem 3.5, we note that any element  $u$  in  $U_{\mathbb{A}}$  can be written as a sum of monomials of the form  $Y_1 Y_2 Y_3 Z X_3 X_2 X_1$ , where the  $X_i$  and  $Y_i$  are as in the theorem and  $Z$  is now in  $U_{\mathbb{A}}^0$ . Let  $w \in M_J^{\mathbb{A}}(\lambda)$ . Then  $w = u \cdot v_\lambda$  for some  $u \in U_{\mathbb{A}}$ . Write  $w = \sum Y_1 Y_2 Y_3 Z X_3 X_2 X_1 \cdot v_\lambda$ . As before, we have  $X_1 \cdot v_\lambda = 0$ , and  $X_2 \cdot v_\lambda = 0$ . Also,  $Z$  commutes with  $X_3$ , up to a scalar in  $\mathbb{A}$ , by Proposition 4.1.

Now we must check the action of  $Z$  on  $v_\lambda$ . First, we have  $K_i^\pm \cdot v_\lambda = q_i^{\pm \lambda(h_i)} v_\lambda \in \mathbb{A} v_\lambda$ , and  $D^\pm \cdot v_\lambda = q_0^{\pm \lambda(d)} v_\lambda \in \mathbb{A} v_\lambda$ . It remains only to check the action of the Lusztig elements. For  $i \in I$ ,  $s \in \mathbb{Z}$  and  $n \in \mathbb{Z}_+$ , we have

$$\begin{bmatrix} K_i & s \\ & n \end{bmatrix} \cdot v_\lambda = \begin{bmatrix} \lambda(h_i) + s \\ & n \end{bmatrix}_{q_i} v_\lambda.$$

The quantum binomials  $\begin{bmatrix} \lambda(h_i) + s \\ n \end{bmatrix}_{q_i}$  are in the ring  $\mathbb{A}$ , and so it follows that

$$\begin{bmatrix} K_i & s \\ & n \end{bmatrix} \cdot v_\lambda \in \mathbb{A} v_\lambda. \text{ Similarly, } \begin{bmatrix} D & s \\ & n \end{bmatrix} \cdot v_\lambda \in \mathbb{A} v_\lambda. \text{ Hence, } Z \cdot v_\lambda \in \mathbb{A} v_\lambda.$$

The remainder of the proof now follows that of Theorem 3.5. We observe that the coefficients of terms of lower degree obtained when commuting basis elements are in  $\mathbb{A}$ . We noted in Section 1 that these coefficients are in fact in  $\mathbb{C}[q, q^{-1}]$ . The result then follows from Theorem 3.5.  $\square$

Now that we have a vector space basis for the  $\mathbb{A}$ -form  $M_J^{\mathbb{A}}(\lambda)$  of  $M_J^q(\lambda)$ , we can begin comparing the two modules, first as vector spaces.

**Proposition 4.3.** *For any  $\lambda \in P$  and  $J \subseteq \dot{I}$ , as  $\mathbb{C}(q)$ -vector spaces,  $\mathbb{C}(q) \otimes_{\mathbb{A}} M_J^{\mathbb{A}}(\lambda) \cong M_J^q(\lambda)$ .*

*Proof.* The proof is fairly standard. The  $\mathbb{C}(q)$ -linear map  $\zeta : \mathbb{C}(q) \otimes_{\mathbb{A}} M_J^{\mathbb{A}}(\lambda) \rightarrow M_J^q(\lambda)$  defined by  $\zeta(f \otimes v) = fv$  for  $f \in \mathbb{C}(q)$  and  $v \in M_J^{\mathbb{A}}(\lambda)$  is clearly surjective. Let  $\{E_{\omega} \cdot v_{\lambda} \mid \omega \in \Omega\}$  be the basis of  $M_J^q(\lambda)$  determined by Theorem 3.5. Let  $\xi : M_J^q(\lambda) \rightarrow \mathbb{C}(q) \otimes_{\mathbb{A}} M_J^{\mathbb{A}}(\lambda)$  be a  $\mathbb{C}(q)$ -linear map defined by

$$\xi(E_{\omega} \cdot v_{\lambda}) = 1 \otimes E_{\omega} \cdot v_{\lambda}.$$

Then, by Proposition 4.2,  $\xi$  is well-defined and the maps  $\zeta$  and  $\xi$  are inverses.  $\square$

We define a weight structure on  $M_J^{\mathbb{A}}(\lambda)$  by setting  $M_J^{\mathbb{A}}(\lambda)_{\mu} = M_J^{\mathbb{A}}(\lambda) \cap M_J^q(\lambda)_{\mu}$  for each  $\mu \in P$ .

**Proposition 4.4.**  *$M_J^{\mathbb{A}}(\lambda)$  is a weight module with the weight decomposition  $M_J^{\mathbb{A}}(\lambda) = \bigoplus_{\mu \in P} M_J^{\mathbb{A}}(\lambda)_{\mu}$ .*

*Proof.* The proof is quite standard, as in [BKMe, Proposition 3.23].  $\square$

The vector-space isomorphism given above restricts to each weight space and we obtain the following result.

**Proposition 4.5.** *For each  $\mu \in P$ ,  $M_J^{\mathbb{A}}(\lambda)_{\mu}$  is a free  $\mathbb{A}$ -module and  $\text{rank}_{\mathbb{A}} M_J^{\mathbb{A}}(\lambda)_{\mu} = \dim_{\mathbb{C}(q)} M_J^q(\lambda)_{\mu}$ .*

## 5. Classical limits.

In this section we take the classical limits of the  $\mathbb{A}$ -forms of the quantum Verma-type modules, and show that they are isomorphic to the Verma-type modules of  $U(\mathfrak{g})$ .

Recall that  $\mathbb{A} = \mathbb{C}[q, q^{-1}, \frac{1}{[n]_{q_i}}, i \in I, n > 0]$ . Let  $\mathbb{J}$  be the ideal of  $\mathbb{A}$  generated by  $q - 1$ . Then there is an isomorphism of fields  $\mathbb{A}/\mathbb{J} \cong \mathbb{C}$  given by  $f + \mathbb{J} \mapsto f(1)$  for any  $f \in \mathbb{A}$ . For any untwisted affine Kac-Moody algebra  $\mathfrak{g}$ , let  $U_{\mathbb{A}} = U_{\mathbb{A}}(\mathfrak{g})$ , and set  $U' = (\mathbb{A}/\mathbb{J}) \otimes_{\mathbb{A}} U_{\mathbb{A}}$ . Then  $U' \cong U_{\mathbb{A}}/\mathbb{J}U_{\mathbb{A}}$ . Denote by  $u'$  the image in  $U'$  of an element  $u \in U_{\mathbb{A}}$ . It was shown by Lusztig [Lu] and DeConcini and Kac [DK] that  $(D')^2 = 1$  and  $(K'_i)^2 = 1$  for all  $i \in I$ . If we let  $K'$  denote the ideal of  $U'$  generated by  $D' - 1$  and  $\{K'_i - 1 \mid i \in I\}$ , then  $\overline{U} = U'/K' \cong U(\mathfrak{g})$ , the universal enveloping algebra of  $\mathfrak{g}$ .

Note that, under the natural map  $U_{\mathbb{A}} \rightarrow U_{\mathbb{A}}/\mathbb{J}U_{\mathbb{A}} \cong U'$ , we have  $q \mapsto 1$ . The composition of natural maps

$$U_{\mathbb{A}} \rightarrow U_{\mathbb{A}}/\mathbb{J}U_{\mathbb{A}} \cong U' \rightarrow \overline{U} = U'/K' \cong U(\mathfrak{g}),$$

is called taking the *classical limit* of  $U_{\mathbb{A}}$ .

Let  $\overline{u} \in \overline{U}$  denote the image of an element  $u \in U_{\mathbb{A}}$ . Then  $\overline{U}$  is generated by the elements  $\overline{E}_i, \overline{F}_i, \overline{D} := \overline{\begin{bmatrix} D & 0 \\ & 1 \end{bmatrix}}$  and  $\overline{H}_i := \overline{\begin{bmatrix} K_i & 0 \\ & 1 \end{bmatrix}}$ ,  $i \in I$ , and, under the isomorphism between  $U(\mathfrak{g})$  and  $\overline{U}$ , the elements  $e_i, f_i, d$  and  $h_i$  may be identified with  $\overline{E}_i, \overline{F}_i, \overline{D}$  and  $\overline{H}_i$ , respectively. Further, Beck [Be1, Section 6] showed that we may identify the  $\overline{E}_{\beta}$  with a PBW basis of  $U(\mathfrak{g})$ , with elements denoted  $e_{\beta}$ .

For  $\lambda \in P$ ,  $J \subseteq \dot{I}$ , let  $M'_J(\lambda) = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M_J^{\mathbb{A}}(\lambda)$ . Then  $M'_J(\lambda) \cong M_J^{\mathbb{A}}(\lambda)/\mathbb{J}M_J^{\mathbb{A}}(\lambda)$  and  $M'_J(\lambda)$  is a  $U'$ -module. For  $\mu \in P$ , let  $M'_J(\lambda)_{\mu} = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M_J^{\mathbb{A}}(\lambda)_{\mu}$ . Since  $M_J^{\mathbb{A}}(\lambda) = \bigoplus_{\mu \in P} M_J^{\mathbb{A}}(\lambda)_{\mu}$ , we must have  $M'_J(\lambda) = \bigoplus_{\mu \in P} M'_J(\lambda)_{\mu}$ . We also have the following standard result.

**Proposition 5.1.** *For  $\mu \in P$ ,  $\dim_{\mathbb{A}/\mathbb{J}} M'_J(\lambda)_\mu = \text{rank}_{\mathbb{A}} M_J^\mathbb{A}(\lambda)_\mu$ .*

*Proof.* By Proposition 4.5, each weight space  $M_J^\mathbb{A}(\lambda)_\mu$ ,  $\mu \in P$ , is a free  $\mathbb{A}$ -module. Let  $\{v_j \mid j \in \Omega\}$  be a basis for  $M_J^\mathbb{A}(\lambda)_\mu$ . Then every element  $v' \in M'_J(\lambda)_\mu = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M_J^\mathbb{A}(\lambda)_\mu$  can be written uniquely as  $v' = \sum_{j \in \Omega} a_j \otimes v_j$  for some scalars  $a_j \in \mathbb{A}/\mathbb{J}$ . (see [Hu, Chapter 4, Theorem 5.11]). Hence,  $\{1 \otimes v_j \mid j \in \Omega\}$  is a basis for  $M'_J(\lambda)_\mu$ .  $\square$

**Proposition 5.2.** *The elements  $D'$  and  $K'_i$  ( $i \in I$ ) in  $U'$  act as the identity on the  $U'$  module  $M'_J(\lambda) = \mathbb{A}/\mathbb{J} \otimes_{\mathbb{A}} M_J^\mathbb{A}(\lambda)$ .*

*Proof.* Let  $\mu \in P$  and  $\{v_j \mid j \in \Omega\}$  be a basis of  $M_J^\mathbb{A}(\lambda)_\mu$ . Then by Proposition 5.1,  $\{v'_j = 1 \otimes v_j \mid j \in \Omega\}$  is an  $\mathbb{A}/\mathbb{J}$ -basis for  $M'_J(\lambda)_\mu$ . Let  $i \in I$ . For each  $j \in \Omega$ , we have  $K_i \cdot v_j = q_i^{\mu(h_i)} v_j$ . Letting  $q \mapsto 1$ , we see  $K'_i \cdot v'_j = v'_j$ . Thus,  $K'_i$  acts on the identity on each weight space of  $M'_J(\lambda)$  and, since  $M'_J(\lambda)$  is a weight module, each  $K'_i$  acts as the identity on the whole space. Similarly,  $D \cdot v_j = q_0^{\mu(d)} v_j$  implies that  $D' \cdot v'_j = v'_j$ , and that  $D'$  acts as a scalar on  $M'_J(\lambda)$ .  $\square$

Since  $M'_J(\lambda)$  is a  $U'$ -module,  $\overline{M}_J(\lambda) = M'_J(\lambda)/K'M'_J(\lambda)$  is a  $\overline{U} = U'/K'$ -module. But  $K'$  was the ideal generated by  $D' - 1$  and the  $K'_i - 1$ , and  $D'$  and each  $K'_i$  acts as the identity on  $M'_J(\lambda)$ , so  $\overline{M}_J(\lambda) = M'_J(\lambda)$ . Since  $\overline{U} \cong U(\mathfrak{g})$ , this means  $\overline{M}_J(\lambda)$  has a  $U(\mathfrak{g})$ -structure. The module  $\overline{M}_J(\lambda)$  is called the classical limit of  $M_J^\mathbb{A}(\lambda)$ . For  $v \in M_J^\mathbb{A}(\lambda)$ , let  $\overline{v}$  denote the image of  $v$  in  $\overline{M}_J(\lambda)$ .

**Proposition 5.3.** *Let  $v_\lambda$  be the generating vector for  $M_J^\mathbb{A}(\lambda)$ . Then as a  $U(\mathfrak{g})$ -module,  $\overline{M}_J(\lambda)$  is a weight module generated by  $\overline{v_\lambda}$  and such that, for any  $\mu \in P$ ,  $\overline{M}_J(\lambda)_\mu$  is the  $\mu$ -weight space of  $\overline{M}_J(\lambda)$ .*

*Proof.* Let  $v_\lambda$  generate  $M_J^\mathbb{A}(\lambda)$ , so that  $M_J^\mathbb{A}(\lambda) = U_{\mathbb{A}} \cdot v_\lambda$ . Then  $\overline{M}_J(\lambda) = \overline{U} \cdot \overline{v_\lambda}$ , so  $\overline{v_\lambda}$  generates  $\overline{M}_J(\lambda)$ . As noted above,  $M'_J(\lambda)$  is a  $U'$ -weight module and since  $\overline{M}_J(\lambda) = M'_J(\lambda)$ ,  $\overline{M}_J(\lambda)$  is also a weight module. Hence,  $\overline{M}_J(\lambda) = \bigoplus_{\mu \in P} \overline{M}_J(\lambda)_\mu$ . It remains to show that the vector space  $\overline{M}_J(\lambda)_\mu$  is actually the  $\mu$ -weight space of  $\overline{M}_J(\lambda)$ . That is, we have to show that  $h_i \cdot \overline{v_\mu} = \mu(h_i) \overline{v_\mu}$  and  $d \cdot \overline{v_\mu} = \mu(d) \overline{v_\mu}$  for all  $i \in I$  and  $\overline{v_\mu} \in \overline{M}_J(\lambda)_\mu$ .

For  $v_\mu \in M_J^\mathbb{A}(\lambda)_\mu$  and  $i \in I$ , we have

$$\begin{bmatrix} K_i & 0 \\ & 1 \end{bmatrix} \cdot v_\mu = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \cdot v_\mu = \frac{q_i^{\mu(h_i)} - q_i^{-\mu(h_i)}}{q_i - q_i^{-1}} v_\mu.$$

Passing to the classical limit, we obtain

$$h_i \cdot \overline{v_\mu} = \overline{H_i} \cdot \overline{v_\mu} = \mu(h_i) \overline{v_\mu}.$$

Similarly, we have  $d \cdot \overline{v_\mu} = \overline{D} \cdot \overline{v_\mu} = \mu(d) \overline{v_\mu}$ .  $\square$

**Proposition 5.4.** *The  $U(\mathfrak{g})$ -module  $\overline{M}_J(\lambda)$  is a  $J$ -highest weight,  $U(\mathfrak{n}_{-J})$ -free module.*

*Proof.* Let  $v_\lambda$  be a generating vector for  $M_J^q(\lambda)$ . By definition,  $E_\beta \cdot v_\lambda = 0$  for all  $\beta \in S_J$ . Hence,  $E_\beta \cdot v_\lambda = 0$  in the  $\mathbb{A}$ -form  $M_J^\mathbb{A}(\lambda)$ . Thus we have  $\overline{E_\beta} \cdot \overline{v_\lambda} = 0$  in  $\overline{M}_J(\lambda)$ . By Proposition 5.3,  $\overline{M}_J(\lambda)$  is a weight module generated by  $\overline{v_\lambda}$  and  $U(\mathfrak{n}_J)$  is spanned by the  $\overline{E_\beta}$ ,  $\beta \in S_J$ , so  $\overline{M}_J(\lambda)$  is a  $J$ -highest weight module.

It remains to show that  $\overline{M}_J(\lambda)$  is a free  $U(\mathfrak{n}_{-J})$ -module. From Proposition 4.2, we know that  $M_J^\mathbb{A}(\lambda)$  is isomorphic to the space spanned by the ordered monomials  $E_{-\beta-n\delta} \cdots E_{-\beta+k\delta} \cdots E_{-\alpha-n\delta} \cdots E_{-k\delta} \cdots E_{\alpha-k\delta}$ , for  $\alpha \in \dot{\Delta}_+ \cap \Delta^J$ ,  $\beta \in \dot{\Delta}_+ \cap \Delta^\infty$ ,  $n \geq 0$ ,  $k > 0$ . Hence,  $\overline{M}_J(\lambda)$  is isomorphic to the space spanned by the ordered monomials  $\overline{E_{-\beta-n\delta}} \cdots \overline{E_{-\beta+k\delta}} \cdots \overline{E_{-\alpha-n\delta}} \cdots \overline{E_{-k\delta}} \cdots \overline{E_{\alpha-k\delta}}$ . But monomials in the images  $\overline{E_{-\beta-n\delta}}$ ,  $\overline{E_{-\beta+k\delta}}$ ,  $\overline{E_{-\alpha-n\delta}}$ ,  $\overline{E_{-k\delta}}$  and  $\overline{E_{\alpha-k\delta}}$  form a basis for  $U(\mathfrak{n}_{-J})$  and so  $\overline{M}_J(\lambda)$  is a  $U(\mathfrak{n}_{-J})$ -free module.  $\square$

We have shown that, for any  $\lambda \in P$ , if we start with a quantum Verma-type module  $M_J^q(\lambda)$ , construct the  $\mathbb{A}$ -form  $M_J^\mathbb{A}(\lambda)$  and take the classical limit, then the resulting module  $\overline{M}_J(\lambda)$  is a  $U(\mathfrak{g})$ -module isomorphic to the Verma-type module  $M_J(\lambda)$ . We have also seen that the weight space structure is preserved under these operations, and so  $M_J^q(\lambda)$  is a true quantum deformation of  $M_J(\lambda)$ .

**Proposition 5.5.** *Let  $\mathfrak{g}$  be an affine Kac-Moody algebra. Let  $\lambda \in P$ ,  $J \subseteq \dot{I}$ . Then the Verma-type module  $M_J(\lambda)$  admits a quantum deformation to the quantum Verma-type module  $M_J^q(\lambda)$  over  $U_q(\mathfrak{g})$  in such a way that the weight space decomposition is preserved. In particular, we have  $\dim M_J^q(\lambda)_\mu \neq 0$  if and only if  $\lambda - \mu$  is in the monoid generated by  $S_J$ ,  $\dim M_J^q(\lambda)_\lambda = 1$ , and  $0 < \dim M_J^q(\lambda)_\mu < \infty$  if and only if  $\lambda - \mu \in Q_+^J$ .*

## 6. Submodule structure of quantum Verma-type modules.

Using the quantum deformation results obtained above, we are now in a position to prove some structural results about quantum modules of Verma-type.

Let  $\mathfrak{g}_q^J$  be the subalgebra of  $U_q(\mathfrak{g})$  generated by monomials of the form  $X_i^{fin}$  and  $Y_i^{fin}$  ( $i = 1, 2, 3$ ) and  $U_q^0(\mathfrak{g})$ . Let  $\tilde{\mathfrak{g}}_q^f$  denote the algebra generated by monomials of the form  $X_1^{fin}, X_3^{fin}, Y_1^{fin}$  and  $Y_3^{fin}$ .

We conjecture that  $\tilde{\mathfrak{g}}_q^f \cong U_q(\mathfrak{g}^f)$ , where  $U_q(\mathfrak{g}^f)$  is the subalgebra of  $U_q(\mathfrak{g})$  generated by  $E_i, F_i$ , ( $i \in J$ ),  $K_i$  ( $i \in I$ ) and  $D$ , but we do not have a proof in general.

Let  $G_q$  be the subalgebra of  $U_q(\mathfrak{g})$  generated by monomials of the form  $X_2$  and  $Y_2$ .

Then the following proposition holds.

**Proposition 6.1.** *Let  $Z$  be the center of  $G_q$ . Then there exists a subalgebra  $\overline{G}_q$  of  $G_q$  such that*

$$\mathfrak{g}_q^J \cong \tilde{\mathfrak{g}}_q^f \otimes_Z \overline{G}_q.$$

*Proof.* Recall the subalgebra  $\overline{G}$  of Proposition 2.2. An element  $x \in \overline{G}$  consists of  $\mathbb{C}$ -linear combinations of imaginary root vectors  $e_{k\delta}^{(i)}$ . Write  $x = \sum c_{i,k} e_{k\delta}^{(i)}$ . For each  $k \in \mathbb{Z}$  and  $i = 1, \dots, N$ , by Beck's construction there is a root vector  $E_{k\delta}^{(i)}$  in  $U_q(\mathfrak{g})$  such that the classical limit of  $E_{k\delta}^{(i)}$  may be identified with  $e_{k\delta}^{(i)}$ . Consider the element  $X = \sum c_{i,k} E_{k\delta}^{(i)}$  in  $U_q(\mathfrak{g})$ . Since the coefficients of the  $E_{k\delta}^{(i)}$ 's are in  $\mathbb{C}$ , certainly  $X \in U_\mathbb{A}$  and the classical limit of  $X$  may be identified with  $x \in \overline{G}$ . Write  $X = \phi(x)$ . Then we define a subalgebra  $\overline{G}_q$  of  $G_q$  by

$$\overline{G}_q = \langle f(q)X \mid f(q) \in \mathbb{C}(q), X = \phi(x) \text{ for some } x \in \overline{G} \rangle.$$

Let  $\tilde{G}_q = G_q \cap \tilde{\mathfrak{g}}_q^f$ . We now show that  $G_q \subseteq \tilde{G}_q \times \overline{G}_q$ . Suppose  $E_{k\delta}^{(i)} \in G_q$ . Then, by Proposition 2.2, we can write  $e_{k\delta}^{(i)} = e + e'$ , where  $e \in \tilde{\mathfrak{g}}^f$  and  $e' \in \overline{G}$ . Let

$E' = \phi(e')$  be the corresponding element in  $\overline{G}_q$ . Since  $e \in \tilde{\mathfrak{g}}^f$ ,  $e$  can be expressed as a linear combination of Lie products of basis vectors  $e_{\alpha+k\delta}$  from root spaces  $\pm\alpha + n\delta$ ,  $\alpha \in \dot{\Delta}^J$ . In  $U_q(\mathfrak{g})$  the corresponding  $E_{\alpha+n\delta}$ 's are all in  $\tilde{\mathfrak{g}}_q^f$  and so  $\mathbb{C}$ -linear combinations of products are in  $\tilde{\mathfrak{g}}_q^f$  as  $\tilde{\mathfrak{g}}_q^f$  is a subalgebra of  $U_q(\mathfrak{g})$ . Hence, there is some  $E$  in  $\tilde{\mathfrak{g}}_q^f$  such that  $E_{k\delta}^{(i)} = E + E'$ . This shows that all the generators of  $\mathfrak{g}_q^J$  can be realized in the product  $\tilde{\mathfrak{g}}_q^f \times \overline{G}_q$ .

Next we show that  $[\tilde{\mathfrak{g}}_q^f, \overline{G}_q] = 0$ . It is enough to show the  $E_{\alpha_j+l\delta}^{(i)}$  ( $j \in J$ ) commute with all  $X = \phi(x)$ ,  $x \in \overline{G}$ . Assume  $X = \sum c_{i,k} E_{k\delta}^{(i)}$ . Then

$$\begin{aligned} [E_{\alpha_j+l\delta}, X] &= [E_{\alpha_j+l\delta}, \sum c_{i,k} E_{k\delta}^{(i)}] \\ &= \sum c_{i,k} [E_{\alpha_j+l\delta}, E_{k\delta}^{(i)}] \\ &= (\pm) \sum c_{i,k} \frac{1}{k} [ka_{ij}]_{d_i} E_{\alpha_j+(k+l)\delta}, \end{aligned}$$

where we obtain the last equality by [BK 1.6.5b], and neglect the sign. All the coefficients of the last right-hand side are in  $\mathbb{A}$ . Taking classical limits, we obtain

$$[e_{\alpha_j+l\delta}, x] = (\pm) \sum c_{i,k} a_{ij} e_{\alpha_j+(k+l)\delta}.$$

But  $[e_{\alpha_j+l\delta}, x] = 0$ , by Proposition 2.2, and  $\sum c_{i,k} a_{ij} e_{\alpha_j+(k+l)\delta} = 0$  if and only if  $\sum c_{i,k} \frac{1}{k} [ka_{ij}]_{d_i} E_{\alpha_j+(k+l)\delta} = 0$ . Hence,  $[E_{\alpha_j+l\delta}, X] = 0$ , and so  $[\tilde{\mathfrak{g}}_q^f, \overline{G}_q] = 0$ . Since  $\tilde{\mathfrak{g}}_q^f \cap \overline{G}_q = Z$ , we conclude that  $\mathfrak{g}_q^J \cong \tilde{\mathfrak{g}}_q^f \otimes_Z \overline{G}_q$ .  $\square$

Let  $M_f^q(\lambda)$  be an ordinary Verma module for  $\mathfrak{g}_q^J$ . Since  $\mathfrak{g}_q^J \cong \tilde{\mathfrak{g}}_q^f \otimes_Z \overline{G}_q$  by Proposition 6.1, we have  $M_f^q(\lambda) \cong \tilde{M}_f^q(\lambda) \otimes \overline{M}^q(\lambda)$ , where  $\tilde{M}_f^q(\lambda)$  is a ‘‘Verma’’ module for  $\tilde{\mathfrak{g}}_q^f$  and  $\overline{M}^q(\lambda)$  is a Verma module for  $\overline{G}_q$ . Further,  $\overline{M}^q(\lambda)$  is irreducible if and only if  $\lambda(c) \neq 0$  as we show below.

Let  $v_\lambda$  be a generating vector for the quantum Verma-type module  $M_f^q(\lambda)$  with  $J$ -highest weight  $\lambda$ .

**Proposition 6.2.** *The  $G_q$ -module  $H^q(\lambda) = G_q \cdot v_\lambda$  is irreducible if and only if  $\lambda(c) \neq 0$ .*

*Proof.* Let  $T_q = \{E_{-\delta}^{(i)} \cdot v_\lambda\}$ , and set  $V_q = G_q \cdot T_q$ . Then  $V_q \neq (0)$  and  $V_q$  is a  $G_q$ -submodule of  $H^q(\lambda) = G_q \cdot v_\lambda$ . Consider the classical limits. We have  $\overline{T} = \{\overline{E_{-\delta}^{(i)}} \cdot \overline{v}_\lambda\} = \{e_{-\delta}^{(i)} \cdot \overline{v}_\lambda\}$  and  $\overline{V} = \overline{G}_q \cdot \overline{T} = G \cdot \overline{T}$ . But  $\overline{V}$  is a non-zero submodule of  $H(\lambda)$ . If  $\lambda(c) = 0$ ,  $\overline{V}$  is a proper submodule of  $H(\lambda)$  and so  $V_q$  is a proper submodule of  $H^q(\lambda)$ . In particular,  $H^q(\lambda)$  is not irreducible.

On the other hand, if  $\lambda(c) \neq 0$ , then  $H(\lambda)$  is irreducible. Suppose  $W$  is a proper submodule of  $H^q(\lambda)$ . Since  $H^q(\lambda) = G_q \cdot v_\lambda$ , using the construction of Proposition 4.3, there is some module  $W_{\mathbb{A}}$  such that  $W \cong \mathbb{C}(q) \otimes_{\mathbb{A}} W_{\mathbb{A}}$ . We refer to this module as the  $\mathbb{A}$ -form of  $W$ . Taking classical limits, we see that  $\overline{W}_{\mathbb{A}}$  is a proper submodule of  $H(\lambda)$ . Hence, we must have  $H^q(\lambda)$  is irreducible if and only if  $\lambda(c) \neq 0$ .  $\square$

**Corollary 6.3.** *The  $\overline{G}_q$ -module  $\overline{M}^q(\lambda)$  is irreducible if and only if  $\lambda(c) \neq 0$ .*

Let  $U_q^{-J}$  be the  $\mathbb{C}(q)$ -linear subspace of  $U_q(\mathfrak{g})$  spanned by monomials of the form  $Y_1^\infty$  and  $X_3^\infty$ . It follows from Theorem 3.5 that  $M_f^q(\lambda) \cong U_q^{-J} \otimes M_f^q(\lambda)$  as vector spaces.

**Theorem 6.4.** *Let  $N$  be a non-trivial submodule of  $M_f^q(\lambda)$ . Then*

- (i)  $N^f := N \cap M_f^q(\lambda) \neq 0$ , and
- (ii) if  $\lambda(c) \neq 0$ , then as vector spaces,  $N \cong U_q^{-J} \otimes N^f$ .

*Proof.* (i) We recall the basis of  $M_f^q(\lambda)$  constructed in Theorem 3.5. Any element of  $M_f^q(\lambda)$  may be written in the form  $x \cdot v_\lambda \in M_f^q(\lambda)$  where  $x$  represents the sum of ordered monomials in this basis. It is enough to consider homogeneous  $x$ . Set  $R(x \cdot v_\lambda)$  to be minus the sum of the heights of the finite roots not in  $\dot{\Delta}^J$  in the decomposition of  $x \cdot v_\lambda$  (i.e., each  $-\alpha + k\delta$  contributes  $\text{ht}(\alpha)$ ) if  $x \cdot v_\lambda \neq 0$  and set  $R(0) = 0$ . Then it is clear that  $R(x \cdot v_\lambda) = 0$  if and only if  $x \in \mathfrak{g}_q^J$ .

It is enough to show that there exists  $y \in U_q(\mathfrak{g})$  such that  $yx \cdot v_\lambda \neq 0$  and  $R(yx \cdot v_\lambda) < R(x \cdot v_\lambda)$ . We will find an element  $y = E_{\alpha-K\delta}$  where  $K$  is sufficiently large and  $\alpha$  is some suitable root.

Let  $w = x \cdot v_\lambda \in M_f^q(\lambda)$ . Then, as in Proposition 4.3, we may write  $w = fw'$  for some  $f \in \mathbb{C}(q)$  and  $w' \in M_J^{\mathbb{A}}(\lambda)$ . Then  $w' = f^{-1}w = f^{-1}x \cdot v_\lambda \in U_q \cdot w$ . Furthermore, suppose  $w' = (q-1)^k w''$ , with  $k > 0$  and  $w'' \in M_J^{\mathbb{A}}(\lambda)$ . Then  $w'' = (q-1)^{-k} w' \in U_q \cdot w$ . Hence, without loss of generality, we may assume  $w = x \cdot v_\lambda \in M_J^{\mathbb{A}}(\lambda)$  and  $q-1$  is not a factor of  $w$  in  $M_J^{\mathbb{A}}(\lambda)$ . Taking the classical limit, we then have  $\bar{w} = \overline{x \cdot v_\lambda} \neq 0$ .

Suppose  $\bar{w}$  is in  $M_f^f(\lambda)$ . Since  $x$  is homogeneous, the grading of  $M_f^q(\lambda)$  ensures that  $w$  is in  $M_f^q(\lambda)$ . Suppose that  $\bar{w}$  is not in  $M_f^f(\lambda)$ . Then by [Fu5, Lemma 5.4] there exists a root  $\alpha$  and nonnegative integer  $K$  such that  $e_{\alpha-K\delta}\bar{w} \neq 0$ . Note that  $e_{\alpha-K\delta}$  is the image of  $E_{\alpha-K\delta}$ . Hence,  $E_{\alpha-K\delta}x \cdot v_\lambda \neq 0$  and  $R(E_{\alpha-K\delta}x \cdot v_\lambda) < R(x \cdot v_\lambda)$ . We complete the proof by induction. This proves part (i) of the theorem.

Let  $0 \neq w \in N$ . As in the proof of part (i), we can assume that  $w \in M_J^{\mathbb{A}}(\lambda)$ . Using the basis of  $M_f^q(\lambda)$ , we may write  $w = \sum u'_i u_i \cdot v_\lambda$ , where  $u'_i \in U_q^{-J}$  and  $u_i \in \mathfrak{n}_{-J}^q$  for each  $i$ . It follows that  $\bar{u}'_i \bar{u}_i \cdot \bar{v}_\lambda \neq 0$  for each  $i$ .

We will assume that each  $u'_i$  is a monomial of type

$$u'_i = E_{-\beta_{i,1} + n_{i,1}\delta}^{l_{i,1}} \cdots E_{-\beta_{s(i),i} + n_{s(i),i}\delta}^{l_{s(i),i}},$$

where  $\beta_{i,j} \in \dot{\Delta}_+ \setminus \Delta^J$ , and  $-\beta_{i,j} + n_{i,j}\delta \neq -\beta_{k,j} + n_{k,j}\delta$  for  $i \neq k$  and  $u'_i \neq u'_j$  for  $i \neq j$ . Further, without loss of generality, we may also assume  $w$  is homogeneous.

Then  $\bar{w} = \sum \bar{u}'_i \bar{u}_i \cdot \bar{v}_\lambda$  and

$$\bar{u}'_i = e_{-\beta_{i,1} + n_{i,1}\delta}^{l_{i,1}} \cdots e_{-\beta_{s(i),i} + n_{s(i),i}\delta}^{l_{s(i),i}}.$$

By [Fu5, Theorem 5.14(i)], we conclude that  $\bar{u}_i \cdot v_\lambda \in \bar{N}^f$  for all  $i$ . Hence,  $u_i \cdot v_\lambda$  belongs to  $N_f^{\mathbb{A}}$ , the  $\mathbb{A}$ -form of  $N$ , and so  $u_i \cdot v_\lambda$  is in  $N_f$  for each  $i$ . The statement follows.  $\square$

**Corollary 6.5.**  *$M_f^q(\lambda)$  is irreducible if and only if  $M_f^q(\lambda)$  is irreducible as  $\mathfrak{g}_q^J$ -module if and only if  $\lambda(c) \neq 0$  and  $\tilde{M}_f^q(\lambda)$  is irreducible as  $\tilde{\mathfrak{g}}_q^f$ -module.*

*Proof.* The results follows from Corollary 6.3, Theorem 6.4(i), and the fact that  $M_f^q(\lambda) \cong \tilde{M}_f^q(\lambda) \otimes \bar{M}^q(\lambda)$ .

## 7. Imaginary Verma modules.

In the special case of imaginary Verma modules we can say rather more about their structure. In particular, we can give a precise description of their submodule structure in the case of level zero. Throughout this section, we shall suppose that  $J = \emptyset$  so that the modules  $M_\emptyset^q(\lambda)$  and  $M_\emptyset(\lambda)$  are imaginary Verma modules for  $U_q(\mathfrak{g})$  and  $U(\mathfrak{g})$ , respectively. We shall also assume that  $\lambda(c) = 0$ . In this case  $H^q(\lambda)$  is a reducible  $G_q$ -module with maximal submodule  $H_\emptyset^q(\lambda)$  consisting of all spaces except  $\mathbb{C}v_\lambda$ .

Denote  $M_{\emptyset,0}^q(\lambda) = U_q(\mathfrak{g})H_\emptyset^q(\lambda)$ . We remark that  $M_{\emptyset,0}^q(\lambda) \neq M_\emptyset^q(\lambda)$ .

Set  $\widetilde{M_\emptyset^q(\lambda)} = M_\emptyset^q(\lambda)/M_{\emptyset,0}^q(\lambda)$ .

**Theorem 7.1.** *The  $U_q(\mathfrak{g})$ -module  $\widetilde{M_\emptyset^q(\lambda)}$  is irreducible if and only if  $\lambda(h_i) \neq 0$ , for all  $i = 1, \dots, N$ .*

*Proof.* Let  $\widetilde{M_\emptyset^q(\lambda)}$  be irreducible and suppose there exists some  $i \in \{1, \dots, N\}$  such that  $\lambda(h_i) = 0$ . Set  $\alpha = \alpha_i$  and  $E = E_{-\alpha}$ . We have that  $\widetilde{M_\emptyset^q(\lambda)} = U_q(\mathfrak{g}) \cdot v_\lambda$ . Consider  $W = U_q E \cdot v_\lambda$ . We show  $W$  is a proper submodule of  $\widetilde{M_\emptyset^q(\lambda)}$ .

Since  $E \cdot v_\lambda \neq 0$ ,  $W \neq (0)$ . Suppose  $W = \widetilde{M_\emptyset^q(\lambda)}$ . Then there must exist elements  $m_j$  in  $U_q$  such that  $v_\lambda = \sum_j m_j E \cdot v_\lambda$ , and for each  $m_j$ , we have  $\text{ht}(m_j) = \alpha$ . (Recall the height of a monomial is the sum of heights of finite roots involved.) Using Beck's ordering and the notation introduced in the proof of Theorem 3.5, we may write each  $m_j$  in the form  $m_j = Y_1 Y_2 Y_3 Z X_3 X_2 X_1$ . We consider the actions of the  $m_j$  on  $E \cdot v_\lambda$ . We need the following lemma.

**Lemma 7.2.** *For any  $k \in \mathbb{Z}$  and  $\beta \in \dot{\Delta}_+$ ,  $E_{-\beta+k\delta} E \cdot v_\lambda = 0$ .*

*Proof.* We divide the proof into cases.

1.  $k \in \mathbb{Z}$ ,  $\beta \neq \alpha$ .

Using Beck's basis, we write  $E_{\beta+k\delta} E \cdot v_\lambda = \sum_i n_i^- n_i^+ v_\lambda$  where  $n_i^+$  are ordered monomials of the form  $X_3 X_2 X_1$  and  $n_i^-$  are ordered monomials of the form  $Y_1 Y_2 Y_3$ . Since  $\beta \neq \alpha$  and  $\alpha$  is simple, then, due to the convexity of the basis, this ordered expression must contain on the right an element of the form  $X_1$  or  $Y_3$  (depending on the sign of  $k$ ). But  $X_1 \cdot v_\lambda = Y_3 \cdot v_\lambda = 0$ . Hence  $E_{\beta+k\delta} E \cdot v_\lambda = 0$ .

2.  $k \in \mathbb{Z} \setminus \{0\}$ ,  $\beta = \alpha$ .

By [BK, (1.6.5d)],  $E_{\alpha+k\delta} E \cdot v_\lambda = X_{k\delta} \cdot \widetilde{v_\lambda} = 0$  for some suitable vector  $X_{k\delta}$  of weight  $k\delta$ , and this has a trivial action in  $\widetilde{M_\emptyset^q(\lambda)}$ .

3.  $k = 0$ ,  $\beta = \alpha$ .

In this case  $E_\alpha E_{-\alpha} = 0$  because  $\lambda(h_i) = 0$ .

In all cases, we have  $E_{-\beta+k\delta} E \cdot v_\lambda = 0$ .  $\square$

### Return to Proof of Theorem

Consider the action of an element  $Y_1 Y_2 Y_3 Z X_3 X_2 X_1 E \cdot v_\lambda$ . By Lemma 7.2,  $X_1 E \cdot v_\lambda = 0$ . Now  $X_2 \cdot v_\lambda = 0$ , and so by [BK, (1.6.5b)],  $X_2 E \cdot v_\lambda$  is in the space of elements of the form  $X_3 \cdot v_\lambda$ . Elements of the form  $Z$  commute with elements of the form  $X_3$  and act as scalars on  $v_\lambda$ . As shown in Theorem 3.5, we may commute elements of the form  $Y_3$  with elements of the form  $X_3$  and then  $Y_3 \cdot v_\lambda = 0$ . Hence we are left considering elements of the form  $Y_1 Y_2 X_3 \cdot v_\lambda$ . The monomial is non-zero as it contains  $E$  and has height 0 as we supposed  $\sum m_j E \cdot v_\lambda = v_\lambda$ . This is not

possible and we have a contradiction. Hence  $W$  is a proper submodule of  $\widetilde{M_\emptyset^q(\lambda)}$  and  $\widetilde{M_\emptyset^q(\lambda)}$  is reducible.

Now we prove the converse statement of the theorem. Let  $\lambda(h_i) \neq 0$ ,  $i = 1, \dots, N$ . Let  $\widetilde{M_\emptyset^\mathbb{A}(\lambda)} = \widetilde{M_\emptyset^q(\lambda)} \cap M_\emptyset^\mathbb{A}(\lambda)$  denote the  $\mathbb{A}$ -form of  $\widetilde{M_\emptyset^q(\lambda)}$ , and let  $\widetilde{M_\emptyset(\lambda)}$  denote its classical limit. Suppose  $N^q$  is a proper submodule of  $\widetilde{M_\emptyset^q(\lambda)}$ . Then  $N^\mathbb{A} = N^q \cap \widetilde{M_\emptyset^\mathbb{A}(\lambda)}$  is a submodule of  $\widetilde{M_\emptyset^\mathbb{A}(\lambda)}$  and the classical limit of  $N^\mathbb{A}$  gives a proper submodule of  $\widetilde{M_\emptyset(\lambda)}$ . But this last module is irreducible by [Fu3]. The theorem is proved.  $\square$

**Proposition 7.3.** *Let  $\lambda(c) = 0$  and  $\lambda(h_i) \neq 0$  for all  $i$ . Then  $M_\emptyset^q(\lambda)$  has an infinite filtration with irreducible quotients of the form  $\widetilde{M_\emptyset^q(\lambda + k\delta)}$ ,  $k \geq 0$ .*

*Proof.* Follows from theorem above.  $\square$

**Corollary 7.4.** *Let  $\lambda(c) = 0$ ,  $\lambda(h_i) \neq 0$ ,  $i = 1, \dots, N$  and  $N^q$  be a submodule of  $M_\emptyset^q(\lambda)$ . Then  $N^q$  is generated by  $N^q \cap H^q(\lambda)$ .*

*Proof.* Obvious from Proposition 7.3.  $\square$

## REFERENCES

- [Be1] J. Beck, *Braid group action and quantum affine algebras*, Commun. Math. Phys. **165** (1994), 555–568.
- [Be2] ———, *Convex bases of PBW type for quantum affine algebras*, Commun. Math. Phys. **165** (1994), 193–199.
- [BK] J. Beck and V.G. Kac, *Finite-dimensional representations of quantum affine algebras at roots of unity*, J. Amer. Math. Soc. **9** (1996), 391–423.
- [BKMe] G. Benkart S.-J. Kang and D.J. Melville, *Quantized enveloping algebras for Borcherds superalgebras*, Trans. Amer. Math. Soc. **350** (1998), 3297–3319.
- [CP] V. Chari and A. Pressley, *A Guide to Quantum groups*, Cambridge University Press, Cambridge, 1994.
- [Co1] B. Cox, *Verma modules induced from nonstandard Borel subalgebras*, Pac. J. Math. **165** (1994), 269–294.
- [Co2] ———, *Structure of nonstandard category of highest weight modules*, Modern Trends in Lie algebra representation theory, V. Futorny, D. Pollack (eds.), 1994, pp. 35–47.
- [CFKM] B. Cox, V.M. Futorny, S.-J. Kang and D.J. Melville, *Quantum deformations of imaginary Verma modules*, Proc. London Math. Soc. **74** (1997), 52–80.
- [CFM] B. Cox, V.M. Futorny and D.J. Melville, *Categories of nonstandard highest weight modules for affine Lie algebras*, Math. Z. **221** (1996), 193–209.
- [Da] I. Damiani, *La R-matrice pour les algèbres quantiques de type affine non tordu*, preprint.
- [DK] C. DeConcini and V.G. Kac, *Representations of quantum groups at roots of 1*, Operator Algebras, Unitary Representations, Enveloping Algebras and Invariant Theory, A. Connes, M. Duflo, A. Joseph, R. Rentschler (eds), Birkhäuser, Boston, 1990, pp. 471–506.
- [Dr] V.G. Drinfel'd, *Hopf algebras and the quantum Yang-Baxter equation*, Soviet Math. Dokl. **32** (1985), 254–258.
- [Fu1] V.M. Futorny, *Root systems, representations and geometries*, Ac. Sci. Ukrain. Math. **8** (1990), 30–39.
- [Fu2] ———, *The parabolic subsets of root systems and corresponding representations of affine Lie algebras*, Contemp. Math. **131** (1992), 45–52.
- [Fu3] ———, *Imaginary Verma modules for affine Lie algebras*, Canad. Math. Bull. **37** (1994), 213–218.

- [Fu4] ———, *Verma type modules of level zero for affine Lie algebras*, Trans. Amer. Math. Soc. **349** (1997), 2663–2685.
- [Fu5] ———, *Representations of affine Lie algebras*, Queen’s Papers in Pure and Applied Mathematics, vol 106, Queen’s University, Kingston, 1997.
- [FGM] V.M. Futorny, A.N. Grishkov and D.J. Melville, *Quantum imaginary Verma modules for affine Lie algebras*, C.R. Math. Rep. Acad. Sci. Canada **20** (1998), 119–123.
- [Ga] F. Gavarini, *A PBW basis for Lusztig’s form on untwisted affine quantum groups*, preprint **q-alg/9709018** (1997).
- [Hu] T. Hungerford, *Algebra*, (5th edition), Springer-Verlag, New York, 1989.
- [JK1] H.P. Jakobsen and V.G. Kac, *A new class of unitarizable highest weight representations of infinite dimensional Lie algebras*, Lecture Notes in Physics, vol. 226, Springer, Berlin, 1985, pp. 1–20.
- [JK2] ———, *A new class of unitarizable highest weight representations of infinite dimensional Lie algebras II*, J. Funct. Anal. **82** (1989), 69–90.
- [Ja] J.C. Jantzen, *Lectures on Quantum Groups*, American Mathematical Society, Providence, 1996.
- [Ji] M. Jimbo, *A  $q$ -difference analogue of  $U(\mathfrak{g})$  and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), 63–69.
- [K] V.G. Kac, *Infinite-dimensional Lie algebras (3rd edition)*, Cambridge University Press, Cambridge, 1990.
- [Ka] S.-J. Kang, *Quantum deformations of generalized Kac-Moody algebras and their modules*, J. Algebra **175** (1995), 1041–1066.
- [KT] S.M. Khoroshkin and V.N. Tolstoy, *On Drinfeld’s realization of quantum affine algebras*, J. Geom. Phys. **11** (1993), 445–452.
- [Le] F.W. Lemire, *Note on weight spaces of irreducible linear representations*, Canad. Math. Bull. **11** (1968), 399–403.
- [Lu] G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras*, Adv. Math. **70** (1988), 237–249.
- [M] D.J. Melville, *An  $\mathbb{A}$ -form technique of quantum deformations*, Recent developments in quantum affine algebras and related topics, Contemp. Math., vol 248, Amer. Math. Soc., 1999, pp. 359–375.

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